Targeted Random Sampling for Reliability Assessment: A Demonstration of Concept

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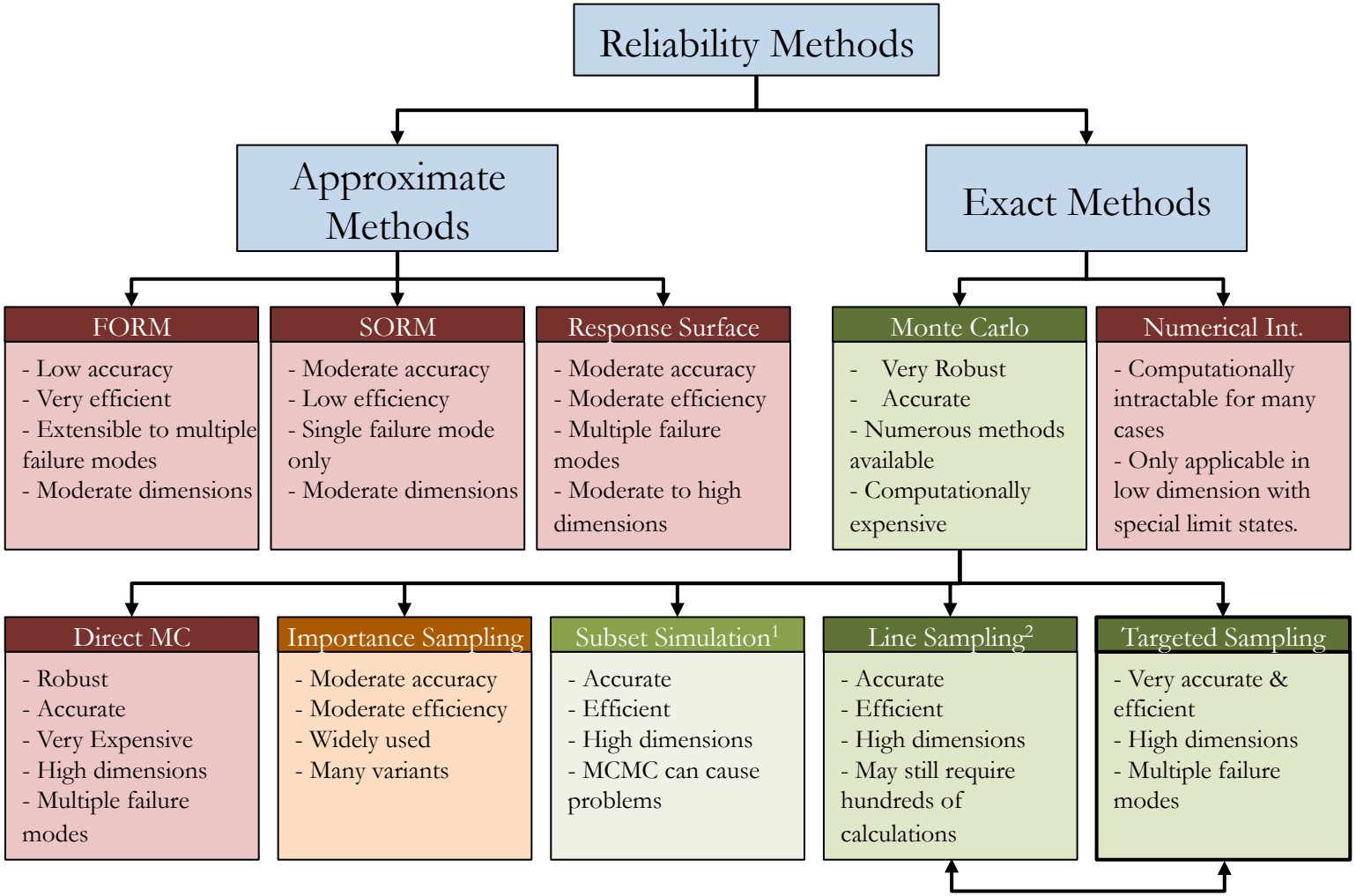
Motivation of Present Work

- Monte Carlo Simulation (MCS) is the most robust and accurate (and sometimes the *only*) means of reliability analysis
- MCS is often (rightfully so) criticized for its high computational cost
- The cost of MCS has been greatly reduced through the advent of sophisticated sampling and simulation methods
- We can still do better!

Outline of Presentation

- Overview of Reliability Methods
  - Emphasis on Monte Carlo Simulation based methods
- Refined Stratified Sampling
  - Concept / Theory
  - Algorithms / Applications
- Targeted Random Sampling
  - Concept / Theory
  - Algorithm
  - Examples
- Challenges & Future Directions

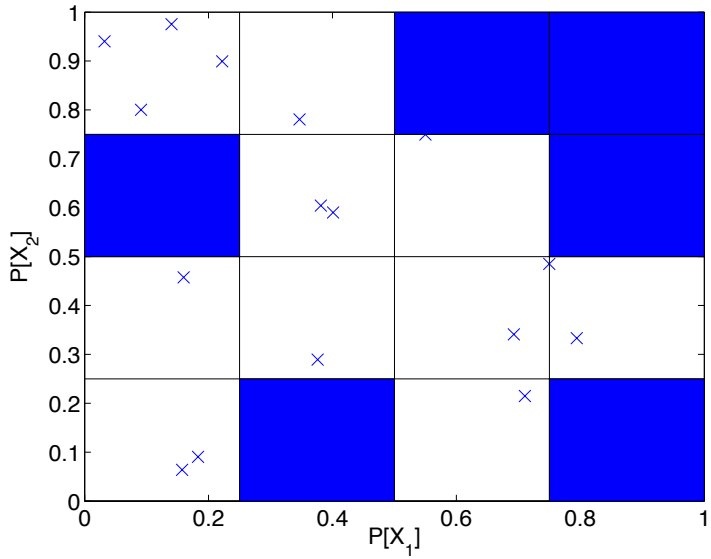
Reliability Analysis



1. Au, S.-K. and Beck, J.L. (2001). "Estimation of small failure probabilities in high dimension by subset simulation" *Probabilistic Engineering Mechanics*
2. Koutsourelakis, P.S., Pradlwarter, H.J., Schueller, G.I. (2004). "Reliability of structures in high dimension, part I: algorithms and applications" *Probabilistic Engineering Mechanics*

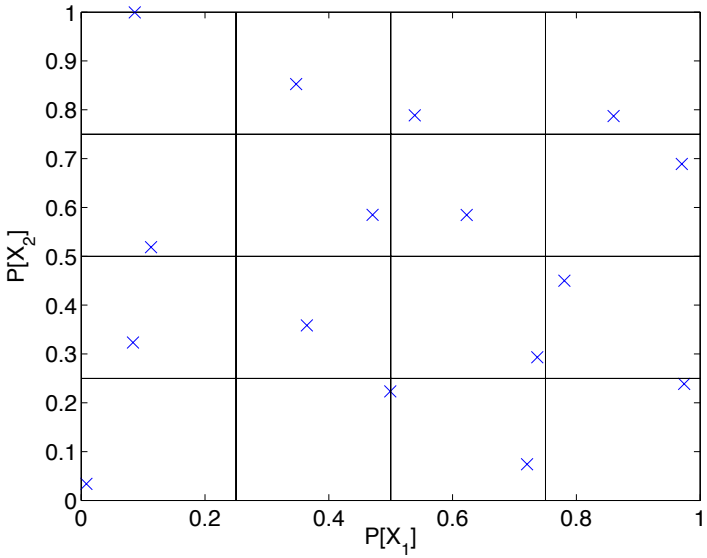
Stratified Sampling

Random Sampling



- Sample size extension is straightforward
- No control over where new samples are placed
- No need to compute sample weights

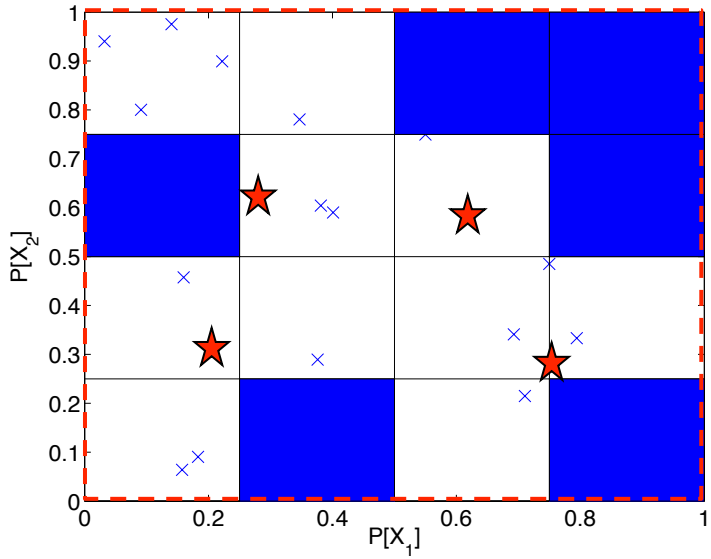
Stratified Sampling



- Almost limitless control over how samples are added
- Carefully select, based on the available information, which strata to add samples to
- Is there a better way?

Stratified Sampling

Random Sampling



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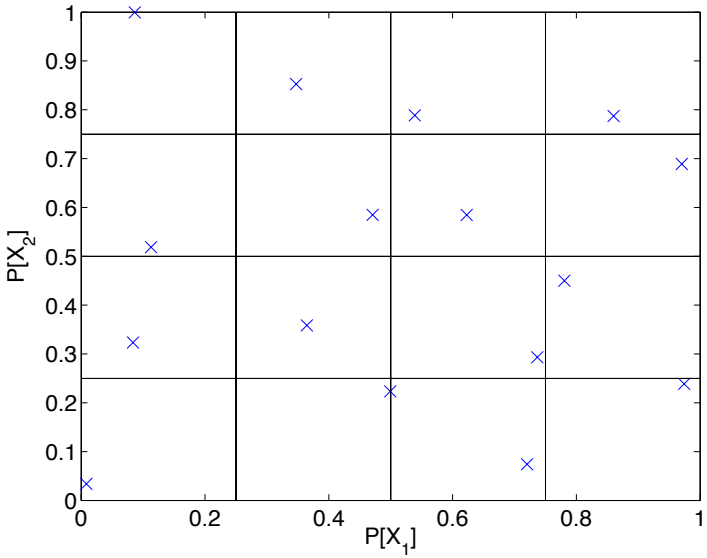

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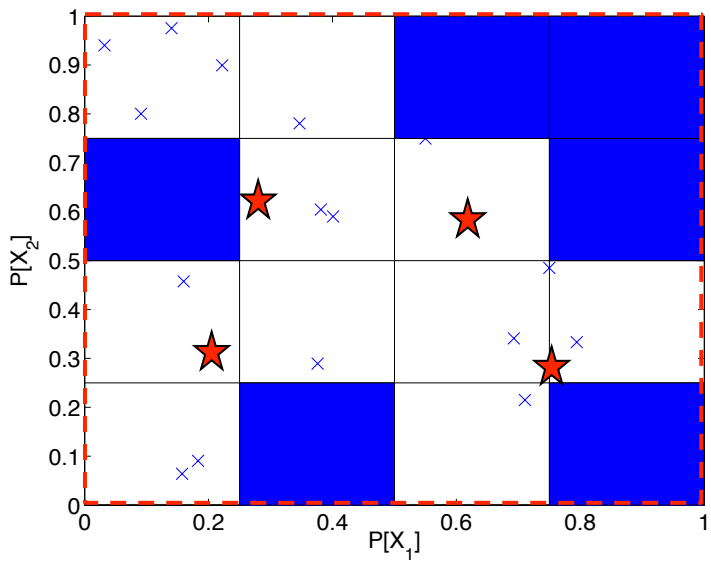

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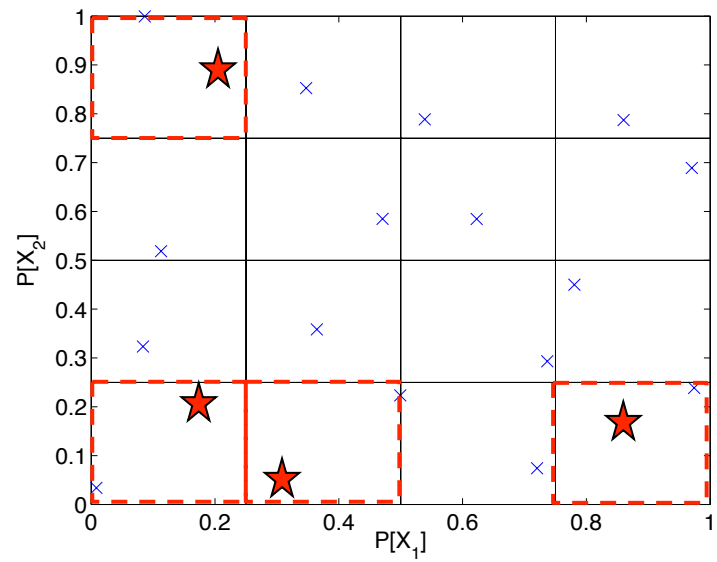

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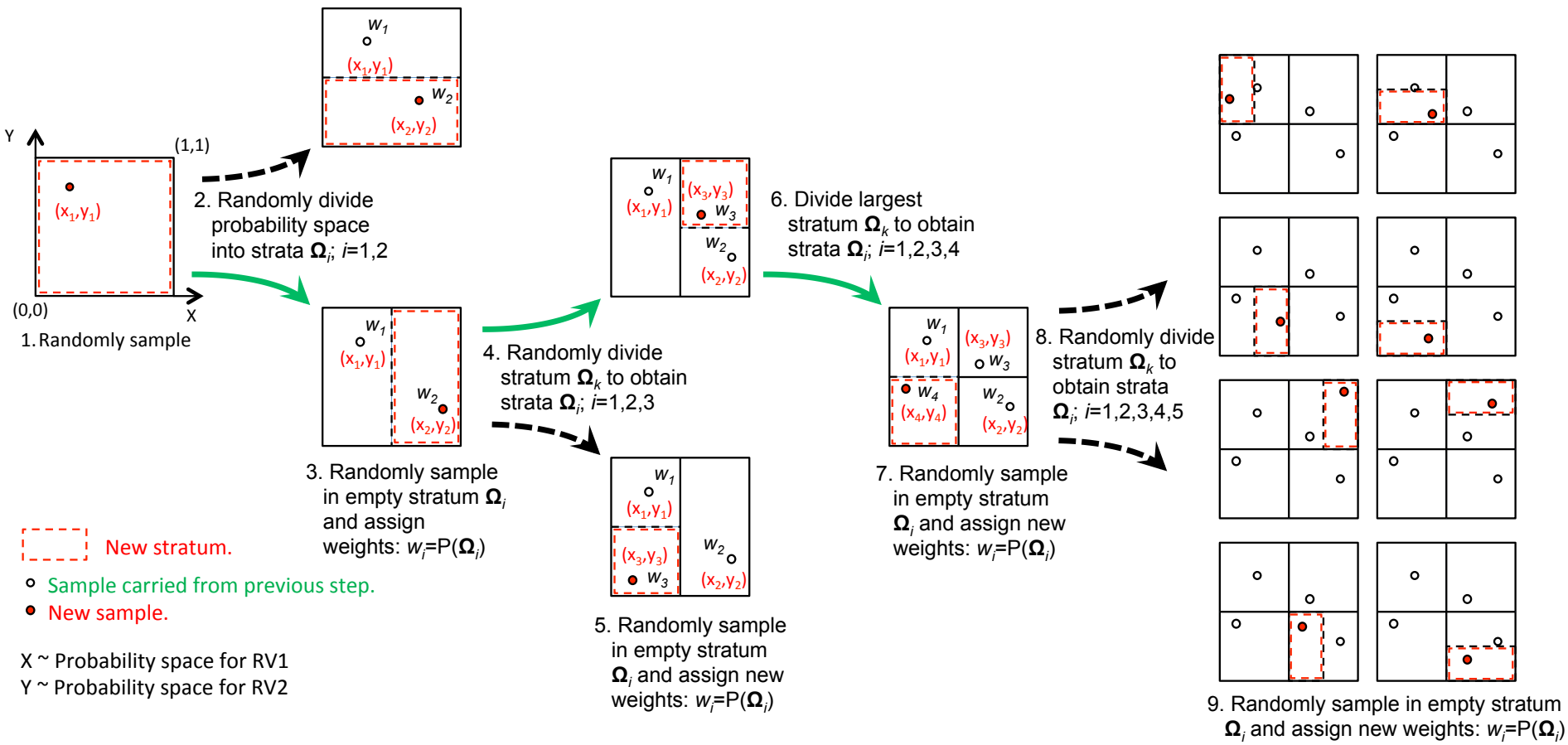
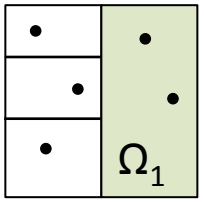
Refined Stratified Sampling



1. Shields, M.D., Teferra, K., Hapij, A. and Daddazio, R. (2014) "Refined stratified sampling for efficient Monte Carlo based uncertainty quantification" *Reliability Engineering & System Safety* (forthcoming)

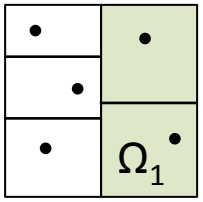
Why Refine Strata?

- Consider a N samples from stratum Ω_1:



$$Var\left[T_S^1\right] = \underbrace{\frac{p_1^2}{N}\sigma_1^2}_{\Omega_1} + \underbrace{\sum_{j=1}^{M}\frac{p_j^2}{n_j}\sigma_j^2}_{\text{Other strata}}$$

- Consider stratum refinement with a single sample in each stratum:



$$Var\left[T_S^2\right] = \underbrace{\frac{p_1^2}{N^2}\sum_{i=1}^{N}\sigma_i^2}_{\Omega_1} + \underbrace{\sum_{j=1}^{M}\frac{p_j^2}{n_j}\sigma_j^2}_{\text{Other strata}}$$

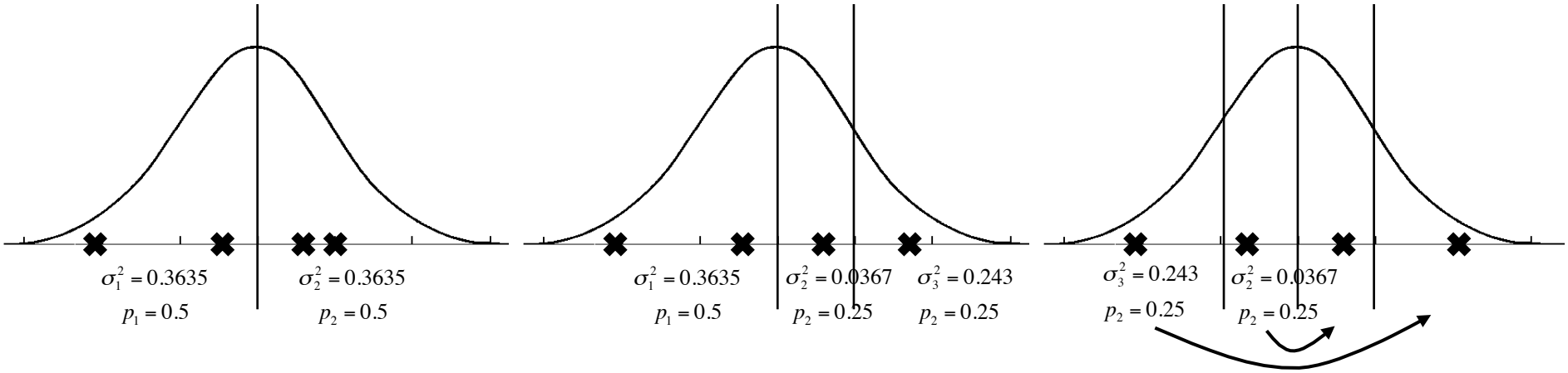
- Evaluate the difference in the variances:

$$Var\left[T_S^1\right] - Var\left[T_S^2\right] = \frac{p_1^2}{N}\left(\sigma_1^2 - \frac{1}{N}\sum_{i=1}^{N}\sigma_i^2\right) > 0$$

(Assumes a "balanced" sub-stratification)

Variance Savings by RSS: Example

- Consider samples drawn from x~N(0,1) with $T_S = E[x]$:



$$Var\left[T_S^b\right] = \frac{\left(\frac{1}{2}\right)^2}{2}\sigma_1^2 + \frac{\left(\frac{1}{2}\right)^2}{2}\sigma_2^2$$

$$Var\left[T_S^b\right] = 0.090875$$

$$Var\left[T_S^u\right] = \frac{\left(\frac{1}{2}\right)^2}{2}\sigma_1^2 + \left(\frac{1}{4}\right)^2\sigma_2^2 + \left(\frac{1}{4}\right)^2\sigma_3^2$$

$$Var\left[T_S^u\right] = 0.0629$$

$$Var\left[T_S^u\right] = \left(\frac{1}{4}\right)^2\sigma_1^2 + \left(\frac{1}{4}\right)^2\sigma_2^2 + \left(\frac{1}{4}\right)^2\sigma_3^2 + \left(\frac{1}{4}\right)^2\sigma_4^2$$

$$Var\left[T_S^u\right] = 0.0349$$

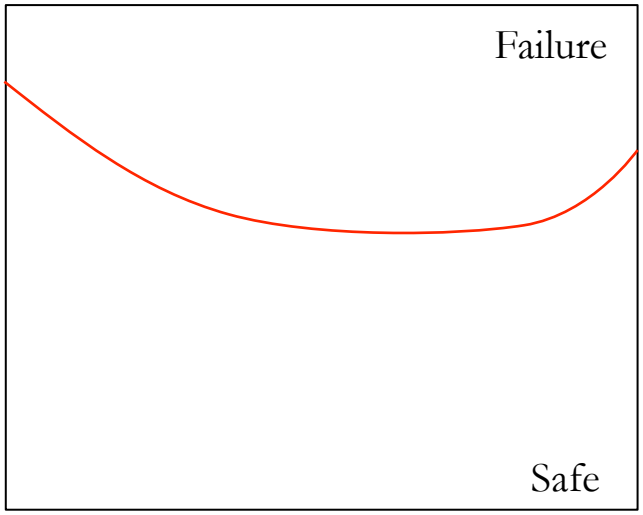
>30% reduction in variance by adding one stratum

>60% reduction in variance by fully stratifying

RSS Algorithms

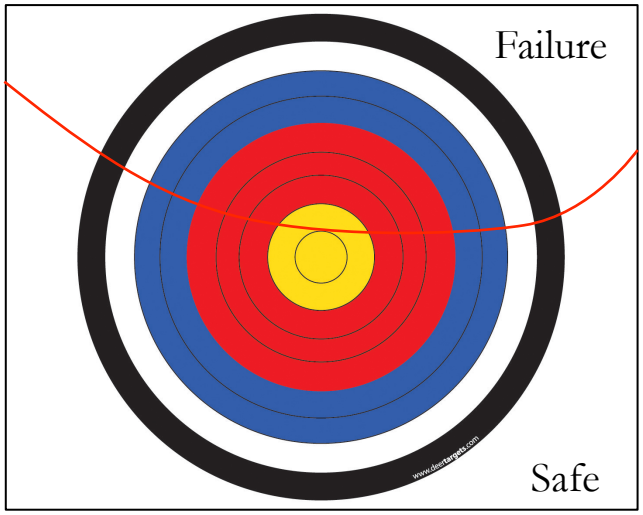
1. Random Stratum Division
2. Variance Minimization
3. Sensitivity-Based Stratum Division
4. Enhanced Space-Filling
5. Targeted Random Sampling
6. Others (Stochastic Search, Stochastic Field Simulation, etc.)

Targeted Random Sampling



Targeted Random Sampling

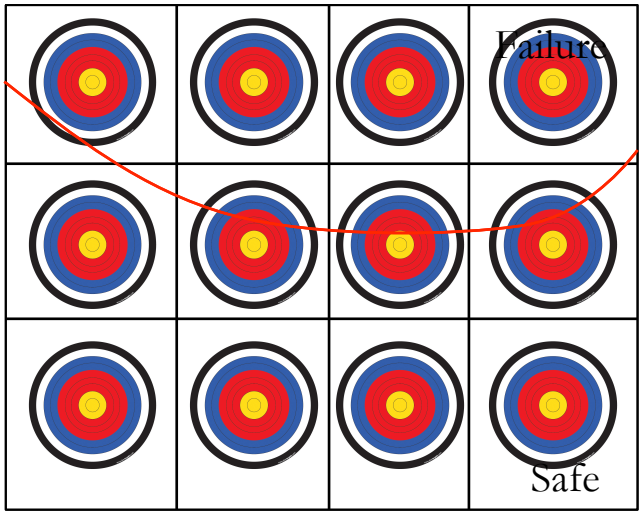
Random (iid) sampling



Targeted Random Sampling

Stratified Sampling/

Refined Stratified Sampling



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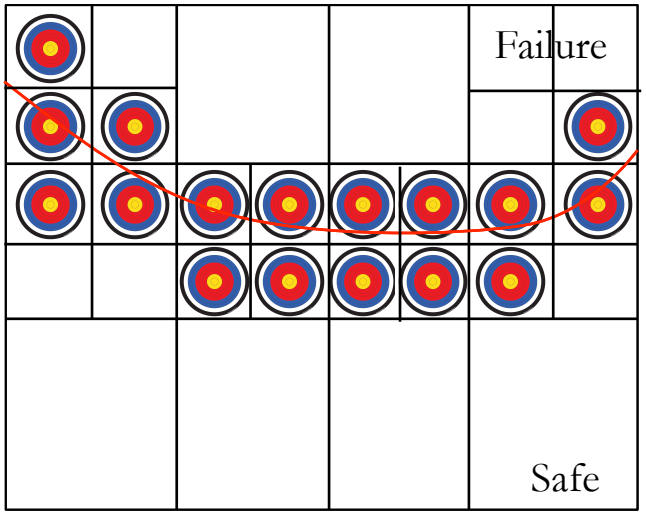


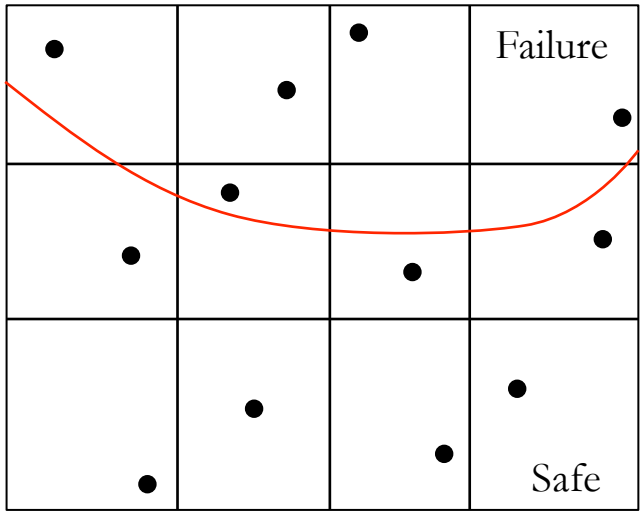
Targeted Random Sampling

The biggest problem with Monte Carlo methods is that they are <u>wasteful</u>

- Many simulations in regions that are, statistically, of little or no use
- This is especially problematic for reliability analysis where probability of failure is <u>very low</u>

Targeted Random Sampling

1. Define an initial stratification of the domain & sample
2. Identify point pairs on opposite sides of the failure surface
3. Identify the pair with the largest separation and divide the stratum at a point in between (alternatively, pair with largest associated probability)
4. Sample in the new stratum
5. Refine the target space
6. Repeat



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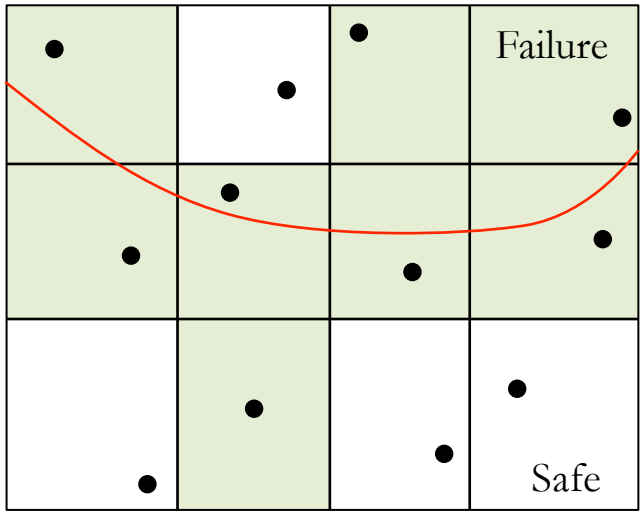
Identify Strata Adjacent to Failure Surface



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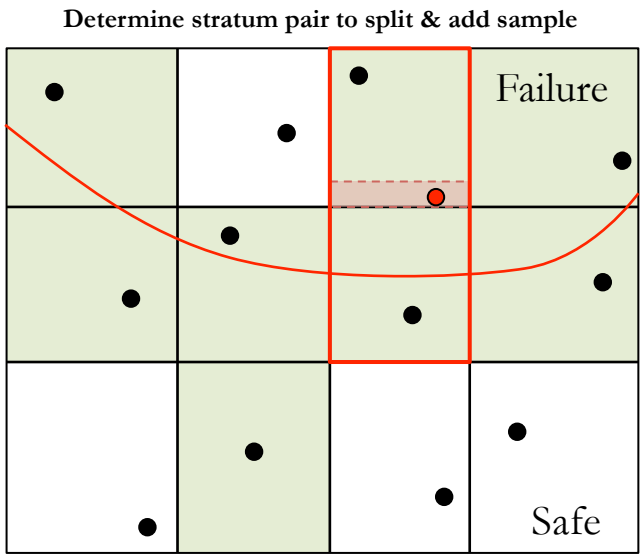

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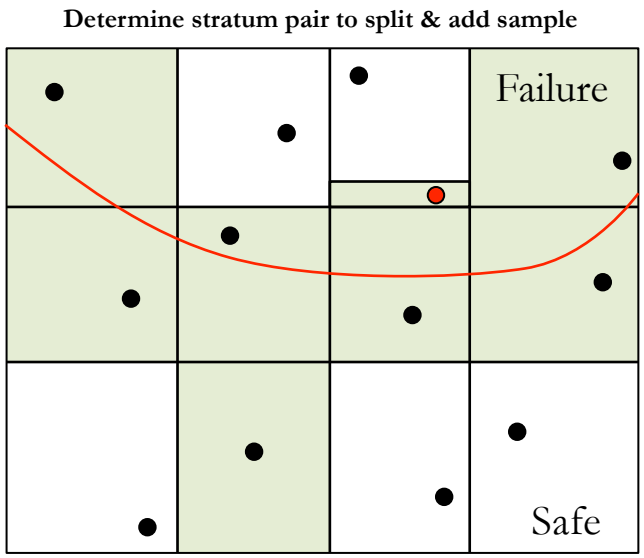

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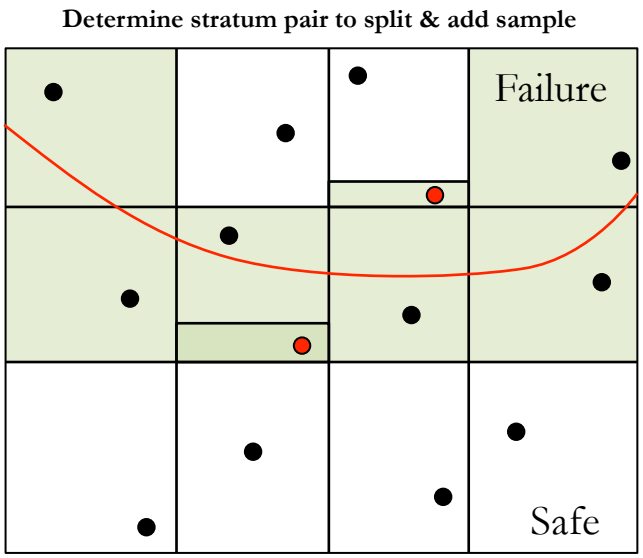

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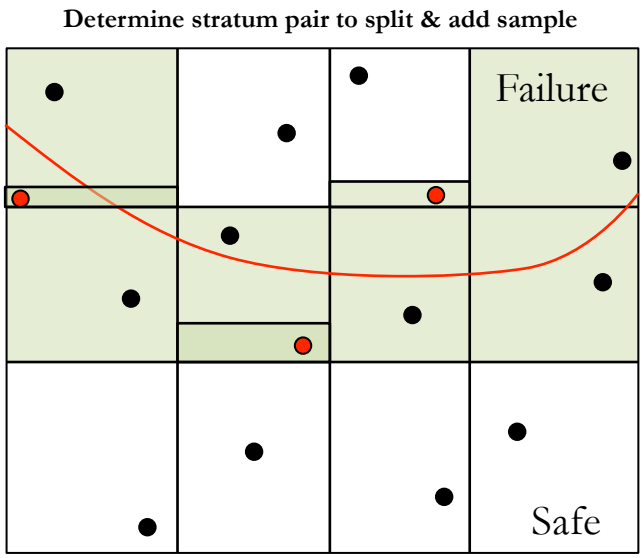

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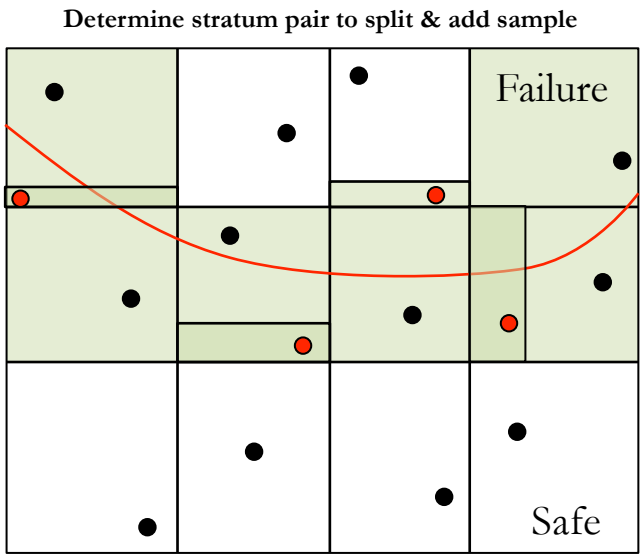

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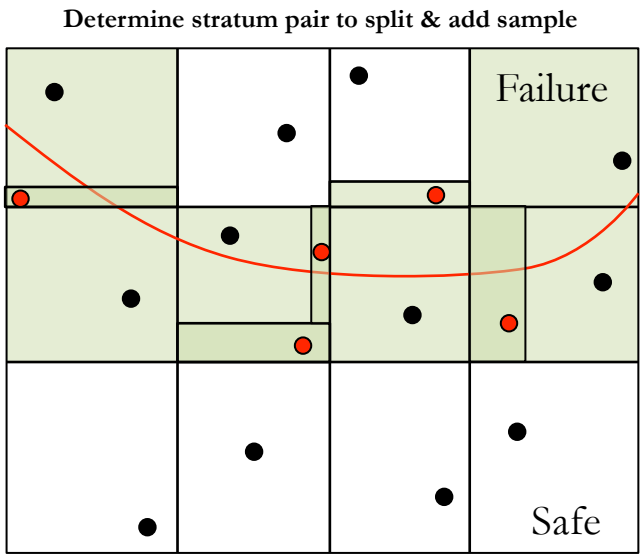

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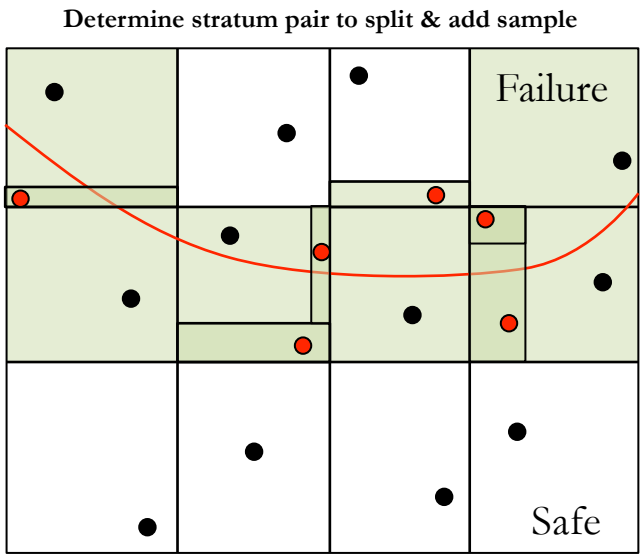

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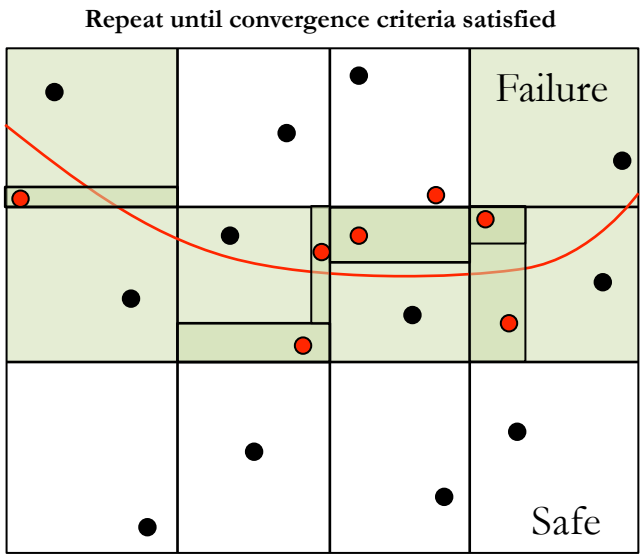

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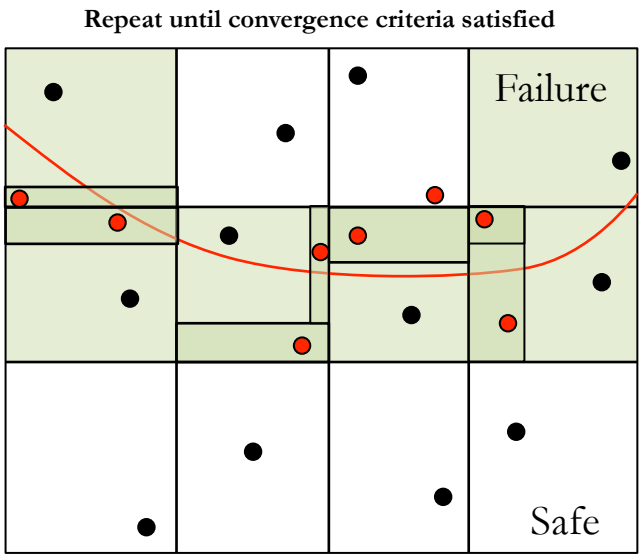

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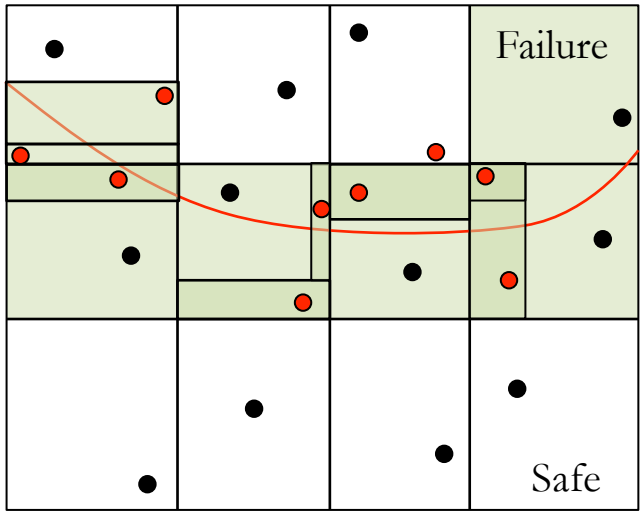
Repeat until convergence criteria satisfied



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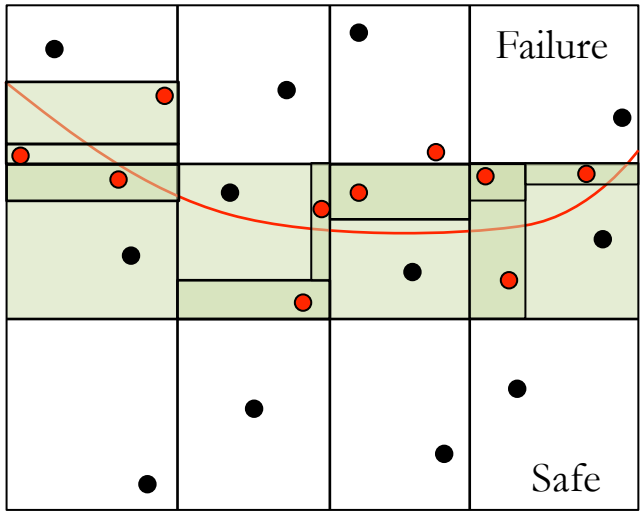

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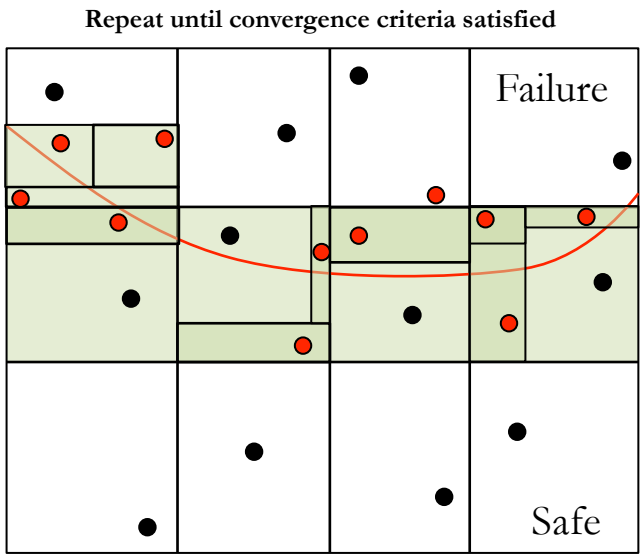

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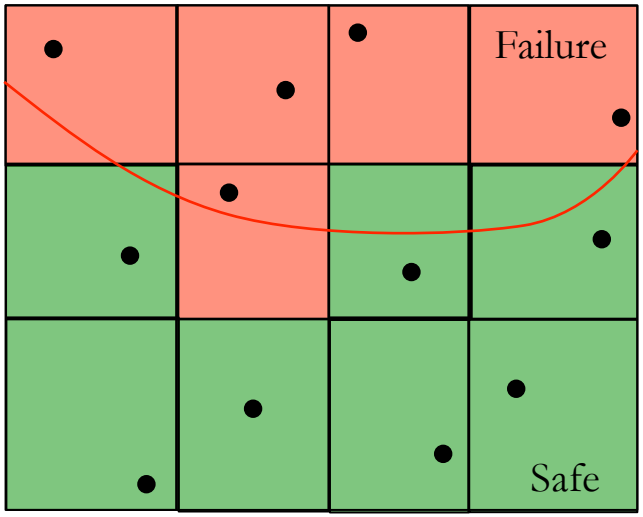

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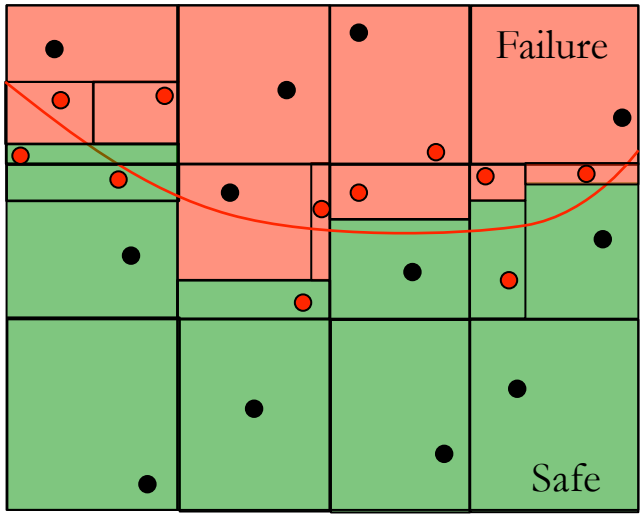
Notes:

- TRS constructs a piecewise approximation of the failure surface
- Need to identify at least one failure point per disjoint failure region
- Uses available information to inform stratification

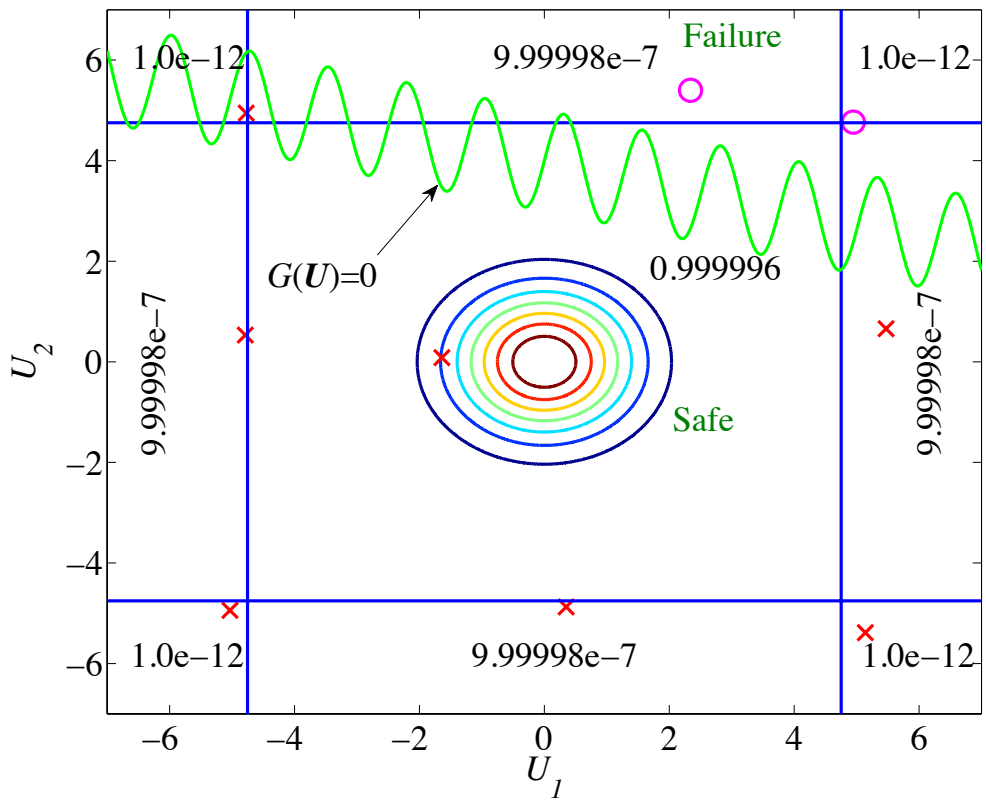
Repeat until convergence criteria satisfied



Example 1: Initial Sample

Limit State: $G(\mathbf{U}) = -\frac{U_1}{4} + \sin\left(5U_1\right) + 4 - U_2; \quad \mathbf{U} \sim N(0,1)$

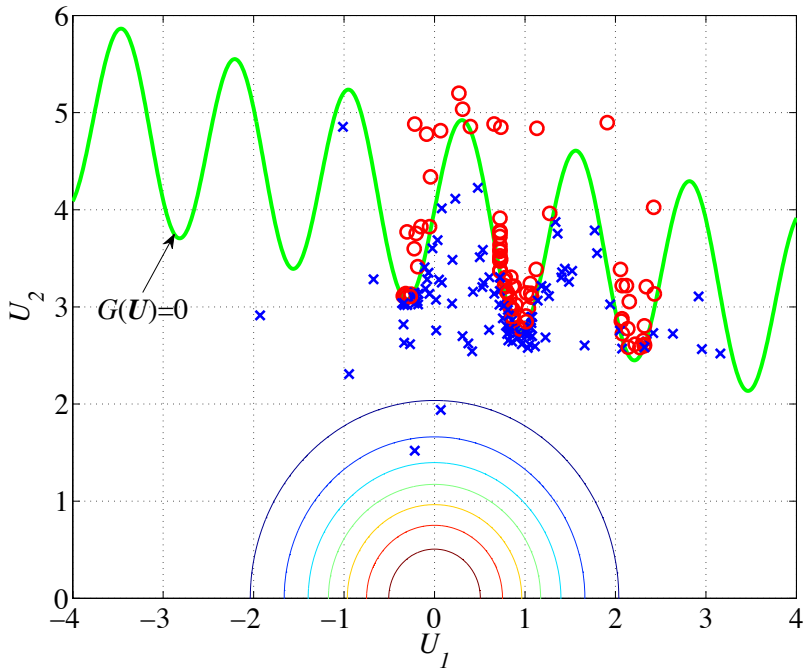
Probability of Failure: $p_f \approx 4.154 \times 10^{-4}$

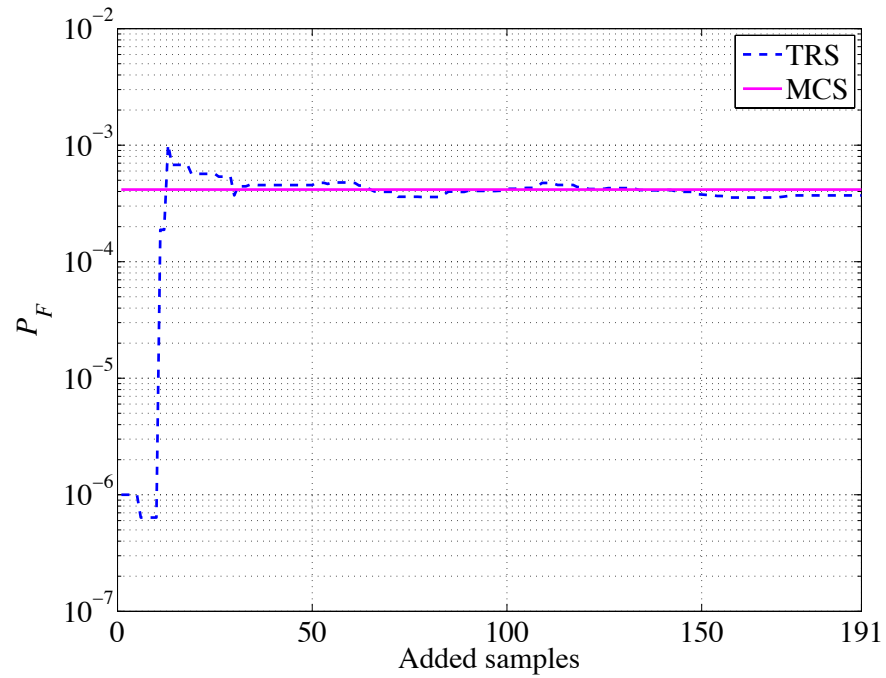


Example 1: 200 Samples

Limit State: $G(\mathbf{U}) = -\frac{U_1}{4} + \sin\left(5U_1\right) + 4 - U_2; \quad \mathbf{U} \sim N(0,1)$

Probability of Failure: $p_f \approx 4.154 \times 10^{-4}$

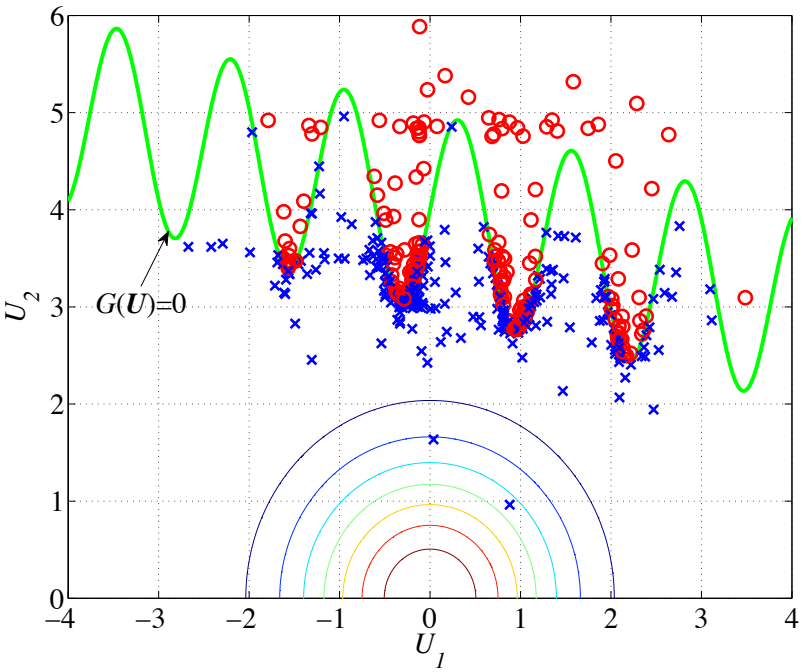


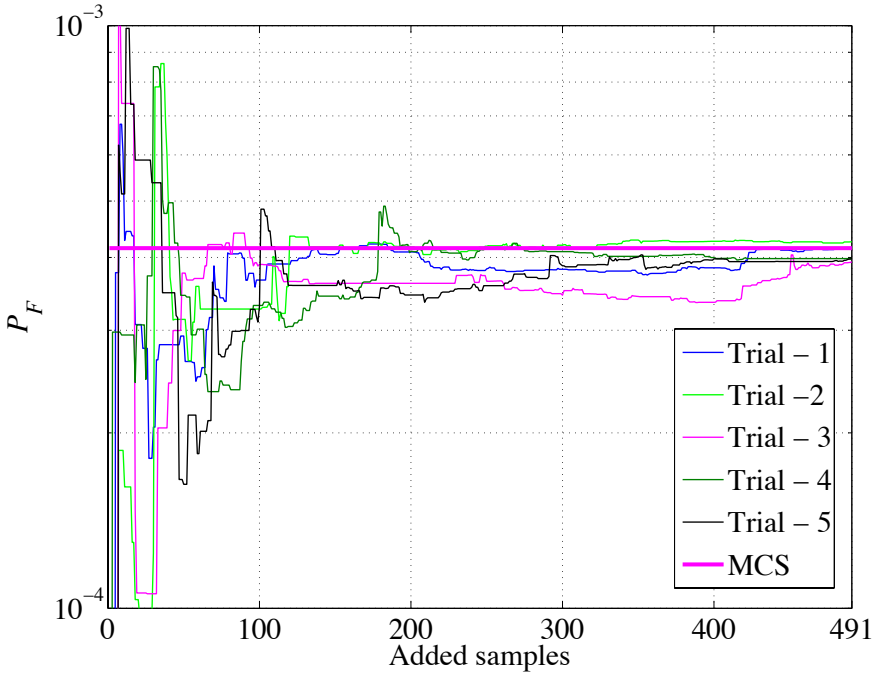


Example 1: 500 Samples

Limit State: $G(\mathbf{U}) = -\frac{U_1}{4} + \sin\left(5U_1\right) + 4 - U_2; \quad \mathbf{U} \sim N(0,1)$

Probability of Failure: $p_f \approx 4.154 \times 10^{-4}$

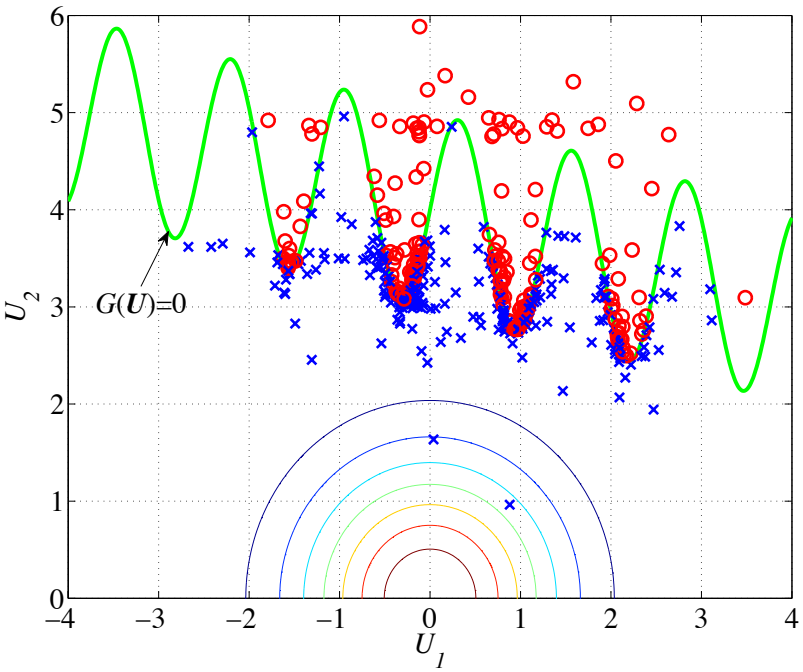




Example 1: Comparison

Limit State: $G(\mathbf{U}) = -\frac{U_1}{4} + \sin\left(5U_1\right) + 4 - U_2; \quad \mathbf{U} \sim N(0,1)$

Probability of Failure: $p_f \approx 4.154 \times 10^{-4}$

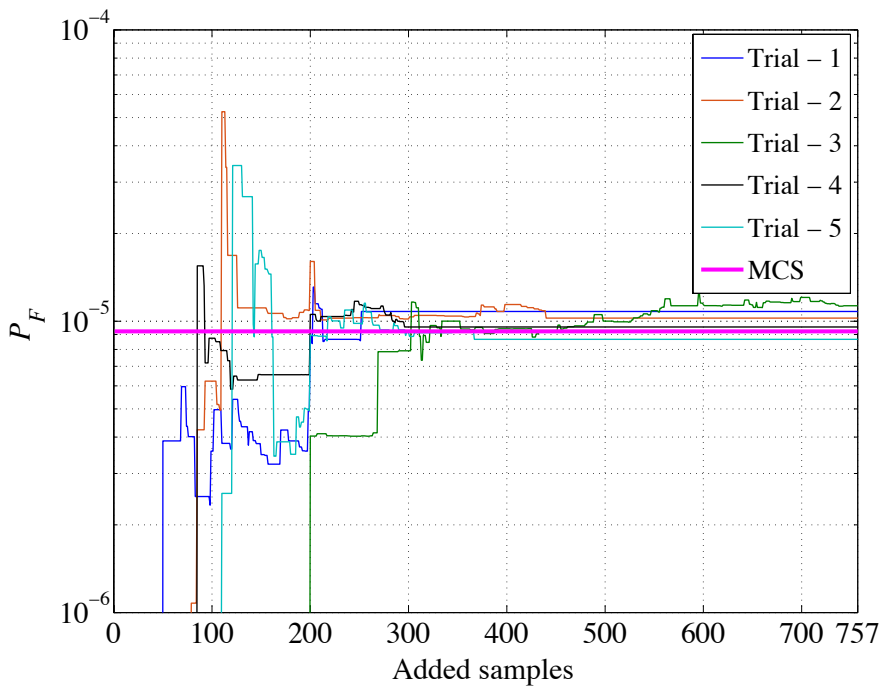


Method	Mean P_F	COV
MCS (10^8 samples)	4.154e-4	
FORM	N/A	
SORM	N/A	
Importance Sampling (IS)	N/A	
SS(500; 50; 10) (~1000 samples)	4.2641e-4 (100)	66%
TRS (9+491) (500 samples)	3.9998e-4 (100)	4%

Example 2: Comparison

Limit State*: $G(\mathbf{X}) = \sum_{i=1}^{5} X_i - C; \quad X_i \sim Exp(\lambda_i); \lambda_i = 1 \, \forall i; C = 0.26715$

Probability of Failure: $p_f \approx 9.2300 \times 10^{-6}$



Method	Mean P_F	COV
MCS (10^8 samples)	9.2300e-6	3%
FORM	1.3913e-4	
SORM	1.4847e-4	
IS (1000 samples)	8.3715e-6	
SS(500; 50; 10) (~1000 samples)	9.8664e-6 (100)	50%
TRS (243+757) (1000 samples)	9.4830e-6 (100)	20%

1. Engelund, S. and Rackwitz, R. (1993). A benchmark study on importance sampling techniques in structural reliability. *Structural Safety*. 12: 255-276.

Challenges & Future Directions

- Defining an appropriate initial stratification in high dimension
  - $3^{20} \sim 3.5e+9$
  - Use MCMC to explore the space and post-stratify
- Dynamic Problems & Stochastic Processes/Fields

- Other RSS designs
  - Variance Minimization for UQ
  - Sensitivity-based Sampling
  - Space-filling sample designs
  - Etc.