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"Reliability Analysis of Structures with Interval Uncertainties under Stochastic Excitations"

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Introduction

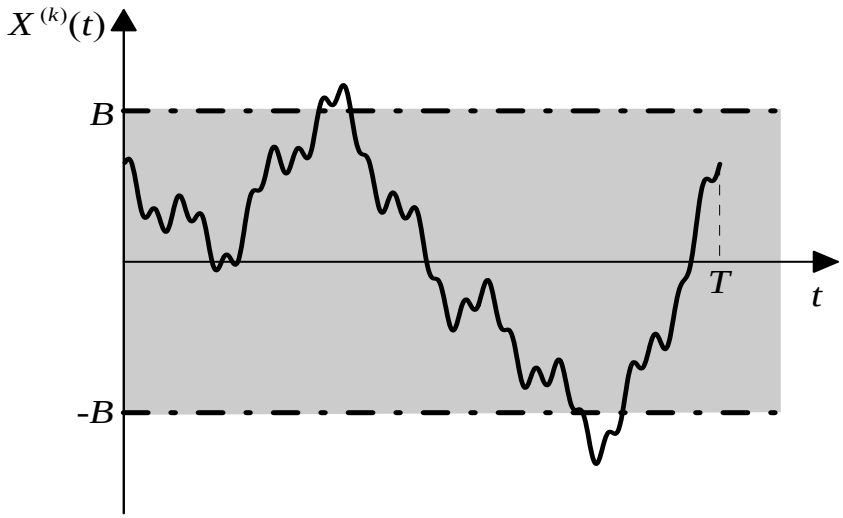
- Uncertainties affecting both structural parameters and external loads need to be included in structural reliability assessment to obtain credible estimates of failure probability.
- However, while the numerous available data permit to model with good accuracy the excitations as stochastic processes, unfortunately the data about the structural parameters are frequently quite limited.
- The credibility of probabilistic reliability methods relies on the availability of sufficient data to describe accurately the probabilistic distribution of the uncertain variables, especially in the tails. Indeed, reliability estimates are very sensitive to small variations of the assumed probabilistic models.
- If available information is fragmentary or incomplete, non-probabilistic approaches, such as convex models, fuzzy set theory or interval models (Ben-Haim ,Elishakoff, 1995; Elishakoff, Ohsaki, 2010), can be alternatively applied for handling structural uncertainties.
- Non-probabilistic methods are complementary rather than competitive to probabilistic methods (Moens, Vandepitte, 2005).

Reliability for stochastic excitations

- Studies on ***reliability analysis*** of randomly excited structures have been carried out mainly introducing the **extreme value process**.

$$X_{\max}(T) = \max_{0 \le t \le T} \left\{ \left| X(t) \right| \right\}$$

- The ***probability of failure*** coincides with the *first passage probability*, i.e. the probability that the *extreme value* random process firstly exceeds the safety bounds within the time interval $[0,T]$.



- The ***reliability function*** represents the probability that the extreme value process is equal to or less than the barrier level $\boldsymbol{B}$ within the time interval $\mathbf{[0,\boldsymbol{T}]}$

$$L_{X_{\max}}(b,T) \equiv \mathcal{P}_s(T,B) = \mathcal{P}\left[\left| X(t) \right| \le B;\, 0 \le t \le T \right] = \mathcal{P}\left[X_{\max}(T) \le B \right]$$

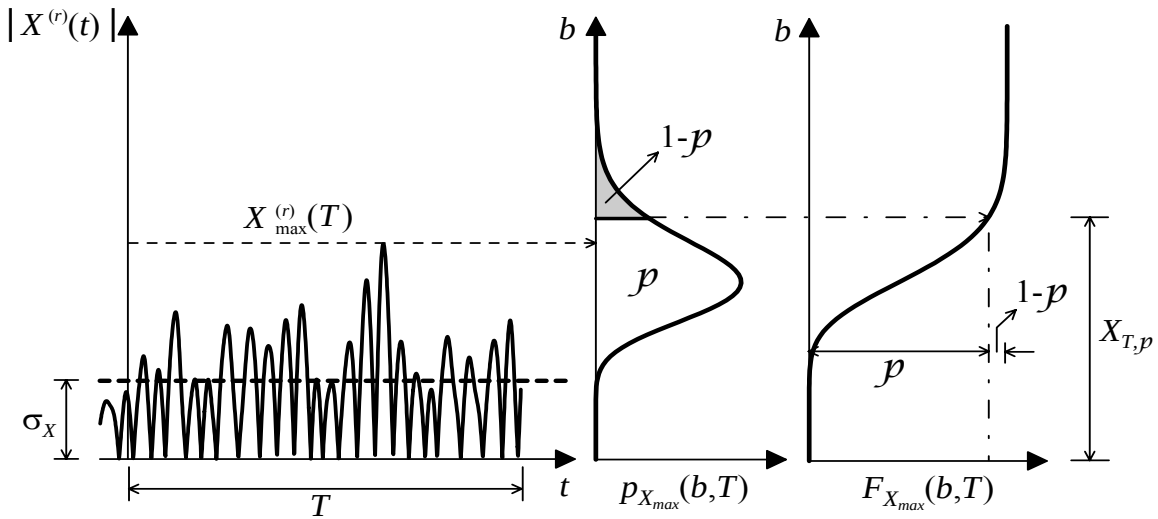
Reliability for stochastic excitations

- Studies on ***reliability analysis*** of randomly excited structures **with deterministic properties** have shown that the reliability function, for zero-mean Gaussian exciting process can be expressed as (Vanmarke,1975):

$$L_{X_{\max}}(b,T) = \mathcal{P}\left[X_{\max}(T) \le b\right] = P_{0,X}(b)\exp\left[-T\,\eta_X(b)\right]$$

$$P_{0,X}(b) = \mathcal{P}[X_{\max}(0) \le b]$$

denotes the initial probability, that is the probability of not exceeding the deterministic level b at time t=0



$$\eta_X(b) = \frac{1}{2\pi}\frac{\sigma_{\dot{X}}}{\sigma_X}\exp\left(-\frac{b^2}{2\sigma_X}\right) = \frac{1}{2\pi}\sqrt{\frac{\lambda_{2,X}}{\lambda_{0,X}}}\exp\left(-\frac{b^2}{2\sqrt{\lambda_{0,X}}}\right)$$

is the so-called *hazard function* (or *limiting decay rate*).

$$\lambda_{\ell,X} = 2\int_0^\infty \omega^\ell\, \mathbf{G}_X(\omega)\,\mathrm{d}\omega$$

spectral moments

(Vanmarke, 1972)

Interval Reliability

- The present contribution deals with the reliability evaluation of linear structural systems with uncertain-but-bounded parameters subjected to stationary Gaussian random excitation.

- Interval reliability evaluation involves zero- and second-order spectral moments of a selected stationary response process: the underlying idea is to derive the interval spectral moments of the stochastic response process and the corresponding interval reliability function in approximate closed-form.

- The ***proposed approach*** relies on:

◇ the use of ***Interval Rational Series Expansion*** (***IRSE***) in conjunction with the ***improved interval analysis*** to obtain an analytical approximation of the interval reliability function.

◇ the derivation of ***interval reliability sensitivities*** with respect to the uncertain parameters by direct differentiation.

◇ the use of ***first-order interval Taylor series expansion*** to obtain estimates of the ***upper and lower bounds of the interval reliability***.

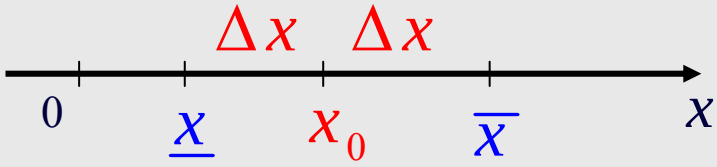
Outline

1. **Improved Interval Analysis**
2. Interval Stochastic Analysis
3. Explicit Interval Reliability Function
4. Numerical Applications
5. Concluding Remarks

Classical Interval Analysis (CIA)

- An **interval number** represents a range of possible values within a closed set:

$$x^I = [x] \triangleq [\underline{x},\ \overline{x}] = \{x \mid \underline{x} \le x \le \overline{x},\ x \in \mathbb{R}\}$$



midpoint: $\boldsymbol{x}_0 = (\underline{x} + \overline{x})/2$,

deviation: $\Delta \boldsymbol{x} = (\overline{x} - \underline{x})/2$.

- **Basic interval operations**: let x^I, y^I and z^I be interval numbers (Moore, 1966)

$$x^I + y^I = [\underline{x} + \underline{y},\ \overline{x} + \overline{y}]; \quad x^I \times y^I = [\min(\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}), \max(\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy})];$$

$$x^I - y^I = [\underline{x} - \overline{y},\ \overline{x} - \underline{y}]; \quad x^I / y^I = [\underline{x},\ \overline{x}] \times [1/\overline{y},\ 1/\underline{y}] \text{ if } 0 \notin y^I.$$

- **Properties of interval arithmetic**:
  - ***Commutative law*:** $x^I + y^I = y^I + x^I;\ x^I \times y^I = y^I \times x^I$
  - ***Associative law*:** $(x^I + y^I) \pm z^I = x^I + (y^I \pm z^I);\ (x^I \times y^I) \times z^I = x^I \times (y^I \times z^I)$
  - ***Subdistributive law*:** $\boxed{x^I \times (y^I + z^I) \subseteq x^I y^I + x^I z^I}$

Furthermore: $\boxed{x^I - x^I \ne 0; \quad x^I / x^I \ne 1}$

Dependency phenomenon

- In the classical interval analysis, the accuracy of the results is affected by the so-called **dependency phenomenon** arising when different occurrences of a single interval variable in an expression are treated as **independent variables**.

- ***For example*** : $f(x^I) = x^I - x^I; \quad x^I = [1,2] \Longrightarrow f(x^I) = [1-2, 2-1] = [-1,1] \neq 0$

  $f(x^I) = x^I(1-1) \Longrightarrow f(x^I) = 0$

- The dependency phenomenon often leads to an **overestimation** of the interval width.

"Reducing the overstimation is a crucial issue to a successful interval analysis"

(Muhanna & Mullen, 2001)

To limit effects of the dependency phenomenon

- ***Affine Arithmetic (AA)*** (Comba & Stolfi,1993; Stolfi & De Figueiredo, 2003)
- ***Parameterized Interval Analysis (PIA)*** (Elishakoff & Miglis, 2012)
- ***Improved Interval Analysis (IIA)*** (Muscolino & Sofi, 2012)

Improved Interval analysis (IIA)

- In ***Affine Arithmetic (AA)*** an interval variable x^I is represented by an *affine form* $\hat{x}^I$, which is a first-degree polynomial:

$$\hat{x}^I = x_0 + x_1\,\varepsilon_1^I + x_2\,\varepsilon_2^I + x_3\,\varepsilon_3^I + \cdots$$

x_0: central value

x_i : partial deviations

$\varepsilon_i^I = [-1,1]$: noise symbols

Each intermediate result is represented by a linear function with a small remainder interval.

- The ***Improved Interval Analysis (IIA)*** is based on the definition of the ***Extra Unitary Interval (EUI)*** $\hat{e}_i^I$

$$\hat{e}_i^I \triangleq [-1,1] \quad \Rightarrow \quad \hat{e}_i^I - \hat{e}_i^I = 0; \qquad \hat{e}_i^I \times \hat{e}_i^I = [1,1]; \qquad \hat{e}_i^I / \hat{e}_i^I = 1$$

$$\alpha^I = \alpha_0 + \Delta\alpha\,\hat{e}_x^I$$

***EUI* enables to treat variables with multiple occurrence as dependent ones**

"Improved Interval Analysis"

$$x_i\hat{e}_i^I \pm y_i\hat{e}_i^I = (x_i \pm y_i)\,\hat{e}_i^I; \qquad x_i\hat{e}_i^I \times y_i\hat{e}_i^I = x_i y_i\left(\hat{e}_i^I\right)^2 = x_i y_i\,[1,1]$$

- "Classical" interval analysis

$$x_i^I + y_i^I = \left[\underline{x}_i + \underline{y}_i,\ \overline{x}_i + \overline{y}_i\right]; \qquad x_i^I - y_i^I = \left[\underline{x}_i - \overline{y}_i,\ \overline{x}_i - \underline{y}_i\right];$$

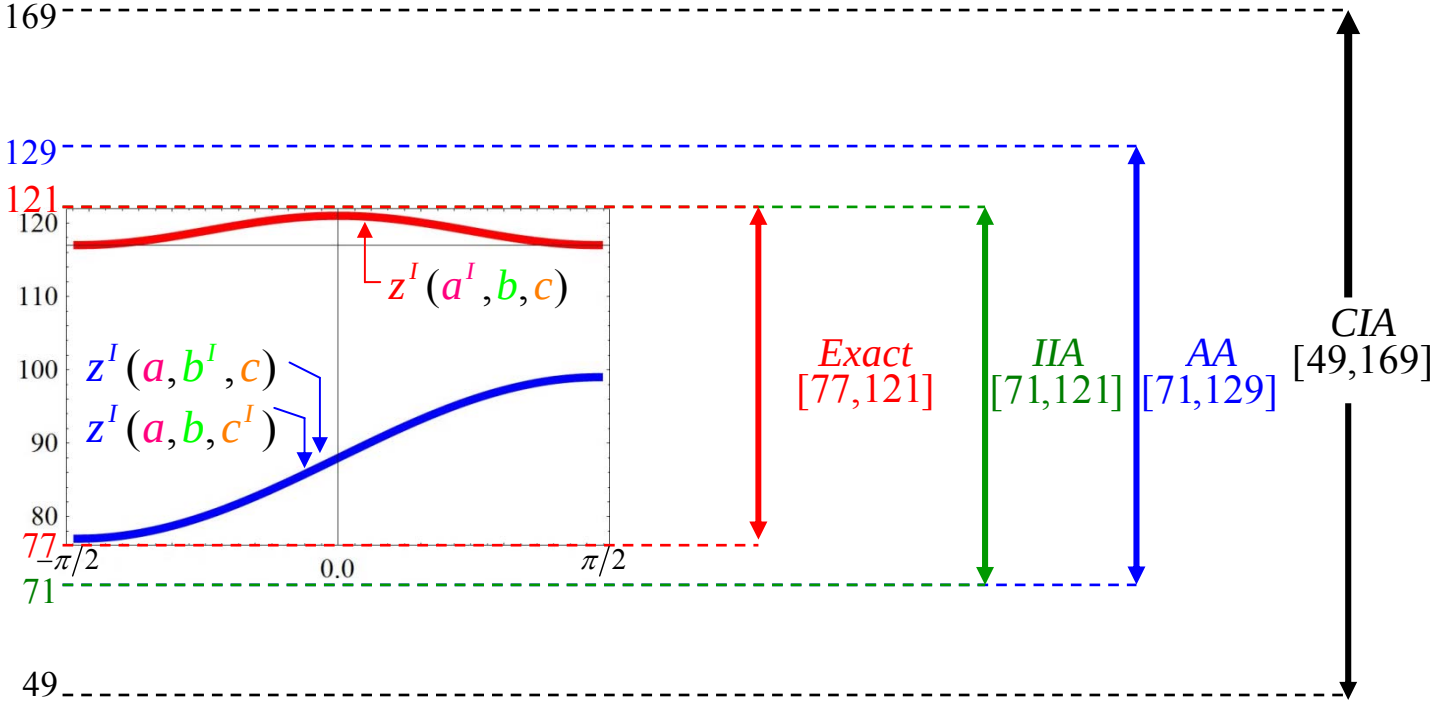
$$x_i^I \times y_i^I = \left[\min\left(\underline{x}_i\underline{y}_i, \underline{x}_i\overline{y}_i, \overline{x}_i\underline{y}_i, \overline{x}_i\overline{y}_i\right), \max\left(\underline{x}_i\underline{y}_i, \underline{x}_i\overline{y}_i, \overline{x}_i\underline{y}_i, \overline{x}_i\overline{y}_i\right)\right];$$

Interval Analysis: an example

- The accuracy of the results obtained by the *Classical Interval Analysis* (***CIA***) , the *Affine Arithmetic* (***AA***) and the *Improved Interval Analysis* (***IIA***) is demonstrated through appropriate comparisons with the *Exact solution.*
- For instance, let us consider the multiplication of two interval functions:

$$z^I = x^I \times y^I = \left(10 + a^I + b^I\right) \times (10 - a^I + c^I)$$

$$a^I = [-2, +2]; \quad b^I = [-1, +1]; \quad c^I = [-1, +1]$$



Interval Rational Series Expansion (IRSE)

- **Interval equilibrium equations in Statics:** $\left(\mathbf{K}(\alpha^I)\right)\mathbf{u}(\alpha^I) = \mathbf{f}$

- **Truss structures** $\alpha^I = \Delta\alpha\, \hat{e}_x^I$

The rank-r change in the stiffness matrix expressed as the superposition of r rank-one matrices:

$$\mathbf{K}(\alpha) = \mathbf{K}_0 + \sum_{i=1}^{r} \Delta\alpha_i \hat{e}_i^I \mathbf{v}_i\, \mathbf{v}_i^T$$

- **Solution of interval equilibrium equations**

$$\mathbf{u}(\alpha^I) = \left(\mathbf{K}(\alpha^I)\right)^{-1}\mathbf{f}; \quad \alpha_i^I = \Delta\alpha_i \hat{e}_i^I \Rightarrow \mathbf{u}(\alpha^I) = \left[\mathbf{K}_0 + \sum_{i=1}^{r} \Delta\alpha_i \hat{e}_i^I \mathbf{v}_i\, \mathbf{v}_i^T\right]^{-1}\mathbf{f}$$

- **Neumann Series Expansion**

$$\mathbf{K}(\alpha)^{-1} = \left[\mathbf{K}_0 + \sum_{i=1}^{r} \Delta\alpha_i \hat{e}_i^I \mathbf{v}_i\, \mathbf{v}_i^T\right]^{-1} = \mathbf{K}_0^{-1} + \sum_{s=1}^{\infty} (-1)^s \left[\mathbf{K}_0^{-1} \sum_{i=1}^{r} \Delta\alpha_i \hat{e}_i^I \mathbf{v}_i\, \mathbf{v}_i^T\right]^s \mathbf{K}_0^{-1}$$

Since this expansion converges very slowly to the exact solution, the so-called ***Interval Rational Series Expansion (IRSE)***, which gives an approximate explicit expression of the inverse of the interval stiffness matrix, has been recently proposed (Muscolino, Santoro & Sofi 2012)

Interval Rational Series Expansion (IRSE)

❑ *IRSE* formula to solve interval equilibrium equations

- The *IRSE* formula can be used to evaluate the inverse of the interval stiffness matrix. In the case of **truss structures**, the *IRSE* yields:

$$\left(\mathbf{K}^I(\boldsymbol{\alpha})\right)^{-1} = \left[\mathbf{K}_0 + \sum_{i=1}^{r} \Delta\alpha_i \hat{e}_i^I \mathbf{v}_i \, \mathbf{v}_i^T\right]^{-1} = \mathbf{K}_0^{-1} - \sum_{i=1}^{r} \frac{\Delta\alpha_i \hat{e}_i^I}{1+\Delta\alpha_i \hat{e}_i^I d_i} \mathbf{D}_i + \sum_{i=1}^{r} \sum_{\substack{j=1 \\ j\neq i}}^{r} \frac{\Delta\alpha_i \hat{e}_i^I \Delta\alpha_j \hat{e}_j^I}{1+\Delta\alpha_j \hat{e}_j^I d_j} d_{ij} \mathbf{D}_{ij}$$

$$- \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{\substack{k=1 \\ k\neq j}}^{r} \frac{\Delta\alpha_i \hat{e}_i^I \Delta\alpha_j \hat{e}_j^I \Delta\alpha_k \hat{e}_k^I}{1+\Delta\alpha_k \hat{e}_k^I d_k} d_{ij} d_{jk} \mathbf{D}_{ik} + \cdots$$

$$d_{is} = \mathbf{v}_i^T \mathbf{K}_0^{-1} \mathbf{v}_s \,; \quad s = j, k, \cdots$$

$$\mathbf{D}_{is} = \mathbf{K}_0^{-1} \mathbf{v}_i \, \mathbf{v}_s^T \mathbf{K}_0^{-1}$$

- If $|\Delta\alpha_i| \ll 1$, the approximate inverse of the interval stiffness matrix can be evaluated by retaining only the first two terms (Impollonia-Muscolino, 2011):

$$\left(\mathbf{K}^I(\boldsymbol{\alpha})\right)^{-1} = \left[\mathbf{K}_0 + \sum_{i=1}^{r} \Delta\alpha_i \hat{e}_i^I \mathbf{v}_i \, \mathbf{v}_i^T\right]^{-1} \approx \mathbf{K}_0^{-1} - \sum_{i=1}^{r} \frac{\Delta\alpha_i \hat{e}_i^I}{1+\Delta\alpha_i \hat{e}_i^I d_i} \mathbf{D}_i$$

The previous formula allows to obtain explicit solutions for the response

It holds if and only if the following conditions are satisfied : $|\alpha_s d_s| < 1; \quad s = i, j, k, \cdots$

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Problem formulation

- Equations of motion of a n-DOF linear structure with uncertain-but-bounded parameters subjected to a stationary multi-correlated Gaussian stochastic process:

$$\boxed{\mathbf{M}_0\,\ddot{\mathbf{U}}(\boldsymbol{\alpha},t)+\mathbf{C}(\boldsymbol{\alpha})\dot{\mathbf{U}}(\boldsymbol{\alpha},t)+\mathbf{K}(\boldsymbol{\alpha})\mathbf{U}(\boldsymbol{\alpha},t)=\mathbf{F}(t),\qquad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right]}$$

- $\boldsymbol{\alpha}\in\mathbb{R}^r$ is the vector collecting the symmetric fluctuations of the uncertain parameters , $\alpha_i^I=\Delta\alpha_i\,\hat{e}_i^I$, $(i=1,\ldots,r)$; $\hat{e}_i^I=[-1,1]$
- $\mathbf{C}(\boldsymbol{\alpha})=c_0\mathbf{M}_0+c_1\mathbf{K}(\boldsymbol{\alpha})$ is the Rayleigh model of the interval damping matrix.
- $\mathbf{F}(t)=\boldsymbol{\mu}_\mathbf{F}+\tilde{\mathbf{X}}_\mathbf{F}(t)$ is fully characterized by the mean-value $\boldsymbol{\mu}_\mathbf{F}$ and PSD $\mathbf{G}_{\tilde{\mathbf{X}}_\mathbf{F}\tilde{\mathbf{X}}_\mathbf{F}}(\omega)$
- Following the interval formalism, the stiffness and damping matrices can be expressed as linear functions of the uncertain physical properties:

$$\boxed{\mathbf{K}(\boldsymbol{\alpha})=\mathbf{K}_0+\sum_{i=1}^{r}\mathbf{K}_i\Delta\alpha_i\,\hat{e}_i^I,\ \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I}\ \Rightarrow\ \boxed{\mathbf{C}(\boldsymbol{\alpha})=\mathbf{C}_0+c_1\sum_{i=1}^{r}\mathbf{K}_i\,\Delta\alpha_i\,\hat{e}_i^I,\ \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I}$$

$$\mathbf{K}_0=\mathbf{K}(\boldsymbol{\alpha}_0);\quad \mathbf{K}_i=\frac{\partial}{\partial\alpha_i}\mathbf{K}(\boldsymbol{\alpha})\bigg|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0};\quad \mathbf{C}_0=c_0\mathbf{M}_0+c_1\mathbf{K}_0$$

Frequency domain stochastic analysis

- Due to the linearity of the system, the stationary Gaussian stochastic response process $\mathbf{U}(\boldsymbol{\alpha},t)$ is characterized from a probabilistic point of view by the definition of the ***mean-value vector*** $\boldsymbol{\mu}_{\mathbf{U}}(\boldsymbol{\alpha})$ and the ***Power Spectral Density (PSD)*** function matrix $\mathbf{G}_{\mathbf{UU}}(\boldsymbol{\alpha},\omega)$, following a frequency domain analysis approach.

- The ***interval response mean-value***, where the input has mean value $\boldsymbol{\mu}_{\mathbf{f}} = \mathrm{E}\langle \mathbf{f}(t) \rangle$, can be determined once the ***inverse of the interval stiffness matrix*** is evaluated, that is:

$$\boxed{\boldsymbol{\mu}_{\mathbf{U}}(\boldsymbol{\alpha}) = \mathrm{E}\langle \mathbf{U}(\boldsymbol{\alpha},t) \rangle = \mathbf{K}^{-1}(\boldsymbol{\alpha})\boldsymbol{\mu}_{\mathbf{F}}, \qquad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}].}$$

- The ***interval response PSD*** function matrix can be determined by

$$\boxed{\mathbf{G}_{\mathbf{UU}}(\boldsymbol{\alpha},\omega) = \mathbf{H}^*(\boldsymbol{\alpha},\omega)\mathbf{G}_{\tilde{\mathbf{X}}_{\mathbf{F}}\tilde{\mathbf{X}}_{\mathbf{F}}}(\omega)\mathbf{H}^T(\boldsymbol{\alpha},\omega), \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}]}$$

requiring the evaluation of the ***interval frequency response function*** (***FRF***) matrix given by:

$$\mathbf{H}(\boldsymbol{\alpha},\omega) = \left[\mathbf{H}_0^{-1}(\omega) + \mathbf{P}(\boldsymbol{\alpha},\omega)\right]^{-1} = \left[\mathbf{I}_n + \mathbf{H}_0(\omega)\mathbf{P}(\boldsymbol{\alpha},\omega)\right]^{-1}\mathbf{H}_0(\omega), \qquad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}].$$

$$\mathbf{H}_0(\omega) = \left[-\omega^2\mathbf{M}_0 + \mathrm{j}\omega\mathbf{C}_0 + \mathbf{K}_0\right]^{-1}$$

$$\mathbf{P}(\boldsymbol{\alpha},\omega) = (1 + \mathrm{j}\omega\, c_1)\sum_{i=1}^{r}\mathbf{K}_i\,\Delta\alpha_i\,\hat{e}_i^I$$

Explicit interval mean-value response vector

- The starting point to derive the ***IRSE*** is the decomposition of the $n \times n$ matrix $\mathbf{K}_i$ as sum of rank-one matrices:

$$\mathbf{K}^I = \mathbf{K}_0 + \sum_{i=1}^{r} \mathbf{K}_i \Delta\alpha_i \hat{e}_i^I = \mathbf{K}_0 + \sum_{i=1}^{r} \mathbf{v}_i \mathbf{v}_i^T \Delta\alpha_i \hat{e}_i^I$$

- Retaining only first-order terms, the ***IRSE*** yields the following ***approximate explicit expression*** of the ***inverse of the interval stiffness matrix*** :

$$\boxed{\left(\mathbf{K}^I\right)^{-1} \approx \mathbf{K}_0^{-1} - \sum_{i=1}^{r} \frac{\Delta\alpha_i \hat{e}_i^I}{1 + \Delta\alpha_i \hat{e}_i^I d_i} \mathbf{D}_i} \qquad \begin{cases} d_i = \mathbf{v}_i^T \mathbf{K}_0^{-1} \mathbf{v}_i ; \\ \mathbf{D}_i = \mathbf{K}_0^{-1} \mathbf{v}_i \mathbf{v}_i^T \mathbf{K}_0^{-1} . \end{cases}$$

- By applying the ***IRSE*** in conjunction with the ***improved interval analysis via EUI***, the ***approximate interval mean-value response vector*** can be expressed as:

$$\boxed{\boldsymbol{\mu}_{\mathbf{U}}^I \equiv \boldsymbol{\mu}_{\mathbf{U}}(\boldsymbol{\alpha}^I) = \left(\mathbf{K}^I\right)^{-1} \boldsymbol{\mu}_{\mathbf{f}} = \mathrm{mid}\left\{\boldsymbol{\mu}_{\mathbf{U}}(\boldsymbol{\alpha}^I)\right\} + \mathrm{dev}\left\{\boldsymbol{\mu}_{\mathbf{U}}(\boldsymbol{\alpha}^I)\right\}}$$

with ***Lower*** and ***Upper*** Bounds:

$$\boxed{\begin{aligned} \underline{\boldsymbol{\mu}}_{\mathbf{U}}(\boldsymbol{\alpha}) &= \mathrm{mid}\left\{\boldsymbol{\mu}_{\mathbf{U}}^I\right\} - \Delta\boldsymbol{\mu}_{\mathbf{U}}(\boldsymbol{\alpha}) \\ \overline{\boldsymbol{\mu}}_{\mathbf{U}}(\boldsymbol{\alpha}) &= \mathrm{mid}\left\{\boldsymbol{\mu}_{\mathbf{U}}^I\right\} + \Delta\boldsymbol{\mu}_{\mathbf{U}}(\boldsymbol{\alpha}) \end{aligned}} \qquad \begin{cases} \mathrm{mid}\left\{\boldsymbol{\mu}_{\mathbf{U}}^I\right\} = \left[\mathbf{K}_0^{-1} + \sum_{i=1}^{r} \tilde{a}_{0,i} \mathbf{D}_i\right] \boldsymbol{\mu}_{\mathbf{f}} ; \quad \tilde{a}_{0,i} = \dfrac{\Delta\alpha_i^2 d_i}{1 - \left(\Delta\alpha_i d_i\right)^2} ; \\ \Delta\boldsymbol{\mu}_{\mathbf{U}}(\boldsymbol{\alpha}) = \sum_{i=1}^{r} \Delta\tilde{a}_i \left|\mathbf{D}_i \boldsymbol{\mu}_{\mathbf{f}}\right| ; \qquad \Delta\tilde{a}_i = \dfrac{\Delta\alpha_i}{1 - \left(\Delta\alpha_i d_i\right)^2} . \end{cases}$$

Explicit interval *FRF* matrix

- Upon substitution of the decomposition $\mathbf{K}_i = \mathbf{v}_i \mathbf{v}_i^T$, the ***interval FRF matrix*** of the structural system with interval parameters takes the following form:

$$\mathbf{H}^I(\omega) = \left[\mathbf{H}_0^{-1}(\omega) + p(\omega) \sum_{i=1}^{r} \mathbf{v}_i \mathbf{v}_i^T \Delta\alpha_i \, \hat{e}_i^I \right]^{-1}$$

with $p(\omega) = (1 + j\omega c_1)$.

- Then, by applying the ***IRSE*** truncated to first-order terms, an ***approximate explicit expression of the interval FRF matrix*** is obtained as:

$$\boxed{\mathbf{H}^I(\omega) \approx \mathbf{H}_0(\omega) - \sum_{i=1}^{r} \frac{p(\omega)\Delta\alpha_i \, \hat{e}_i^I}{1 + p(\omega)\Delta\alpha_i \, \hat{e}_i^I \, b_i(\omega)} \mathbf{B}_i(\omega)}$$

$$b_i(\omega) = \mathbf{v}_i^T \mathbf{H}_0(\omega) \mathbf{v}_i \, ;$$

$$\mathbf{B}_i(\omega) = \mathbf{H}_0(\omega) \mathbf{v}_i \mathbf{v}_i^T \mathbf{H}_0(\omega).$$

- Alternatively, by applying the ***improved interval analysis via EUI:***

$$\mathbf{H}(\boldsymbol{\alpha}, \omega) = \mathrm{mid}\left\{ \mathbf{H}^I(\omega) \right\} + \mathrm{dev}\left\{ \mathbf{H}^I(\omega) \right\}$$

$$\mathbf{H}_{\mathrm{mid}}(\boldsymbol{\alpha}, \omega) = \mathbf{H}_0(\omega) + \sum_{i=1}^{r} a_{0,i}(\omega) \mathbf{B}_i(\omega)$$

$$\mathbf{H}_{\mathrm{dev}}(\omega) = \sum_{i=1}^{r} \Delta a_i(\omega) \mathbf{B}_i(\omega) \hat{e}_i^I$$

with $a_{0,i}(\omega) = \dfrac{\left[p(\omega)\Delta\alpha_i \right]^2 b_i(\omega)}{1 - \left[p(\omega)\Delta\alpha_i \, b_i(\omega) \right]^2}$; $\Delta a_i(\omega) = \dfrac{p(\omega)\Delta\alpha_i}{1 - \left[p(\omega)\Delta\alpha_i \, b_i(\omega) \right]^2}$.

Explicit Interval *PSD* matrix

- Substituting the interval transfer function matrix, the ***interval PSD function matrix*** of the structural response can be expressed as :

$$\mathbf{G}_{\mathbf{UU}}(\boldsymbol{\alpha},\omega) \equiv \mathbf{G}^{I}_{\mathbf{UU}}(\omega) = \mathbf{H}^{*}(\boldsymbol{\alpha},\omega)\,\mathbf{G}_{\tilde{\mathbf{X}}_{\mathbf{f}}\tilde{\mathbf{X}}_{\mathbf{f}}}(\omega)\,\mathbf{H}^{T}(\boldsymbol{\alpha},\omega)$$
$$= \mathrm{mid}\left\{\mathbf{G}_{\mathbf{UU}}(\boldsymbol{\alpha},\omega)\right\} + \mathrm{dev}\left\{\hat{\mathbf{G}}_{\mathbf{UU}}(\boldsymbol{\alpha},\omega)\right\}$$

where

$$\mathrm{mid}\left\{\mathbf{G}^{I}_{\mathbf{UU}}(\omega)\right\} = \mathbf{H}^{*}_{\mathrm{mid}}(\boldsymbol{\alpha},\omega)\,\mathbf{G}_{\tilde{\mathbf{X}}_{\mathbf{f}}\tilde{\mathbf{X}}_{\mathbf{f}}}(\omega)\,\mathbf{H}^{T}_{\mathrm{mid}}(\boldsymbol{\alpha},\omega);$$

$$\mathrm{dev}\left\{\hat{\mathbf{G}}^{I}_{\mathbf{UU}}(\omega)\right\} = \mathbf{H}^{*}_{\mathrm{mid}}(\boldsymbol{\alpha},\omega)\,\mathbf{G}_{\tilde{\mathbf{X}}_{\mathbf{f}}\tilde{\mathbf{X}}_{\mathbf{f}}}(\omega)\left(\mathbf{H}^{I}_{\mathrm{dev}}(\omega)\right)^{T} + \left(\mathbf{H}^{I}_{\mathrm{dev}}(\omega)\right)^{*}\mathbf{G}_{\tilde{\mathbf{X}}_{\mathbf{f}}\tilde{\mathbf{X}}_{\mathbf{f}}}(\omega)\,\mathbf{H}^{T}_{\mathrm{mid}}(\boldsymbol{\alpha},\omega)$$

- Finally, approximate analytical expressions of the ***interval spectral moments*** of order ℓ of the random response, useful for structural reliability, can be computed as:

$$\boldsymbol{\lambda}_{\ell,\mathbf{UU}}(\boldsymbol{\alpha}) = \mathrm{mid}\left\{\boldsymbol{\lambda}_{\ell,\mathbf{UU}}(\boldsymbol{\alpha},\omega)\right\} + \mathrm{dev}\left\{\boldsymbol{\lambda}_{\ell,\mathbf{UU}}(\boldsymbol{\alpha},\omega)\right\}, \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^{I} = \left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right]; \quad \ell = 0,1,2$$

where $\mathrm{mid}\left\{\boldsymbol{\lambda}_{\ell,\mathbf{UU}}\right\} = 2\int_{0}^{\infty}\omega^{\ell}\,\mathrm{mid}\left\{\mathbf{G}^{I}_{\mathbf{UU}}(\omega)\right\}\mathrm{d}\omega; \quad \mathrm{dev}\left\{\boldsymbol{\lambda}_{\ell,\mathbf{UU}}\right\} = 2\int_{0}^{\infty}\omega^{\ell}\,\mathrm{dev}\left\{\hat{\mathbf{G}}^{I}_{\mathbf{UU}}(\omega)\right\}\mathrm{d}\omega.$

➢ It is worth to emphasize that explicit relationships between the interval statistics of the displacement vector $\mathbf{U}(\boldsymbol{\alpha},t)$ and the radius $\Delta\alpha_i$ of input parameters, useful for the analytical evaluation of the corresponding ***interval sensitivities,*** have been provided.

Outline

1. Improved Interval Analysis
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3. **Explicit Interval Reliability Function**
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Interval Reliability Function

- For a structure with uncertain-but-bounded parameters, the ***extreme value random process***, over a specified time interval $[0,T]$, is mathematically defined as:

$$\boxed{U_{\max}(\boldsymbol{\alpha},T)=\max_{0\le t\le T}\left|U(\boldsymbol{\alpha},t)\right|,\quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right]}$$

- The ***cumulative distribution function (CDF)*** $L_{U_{\max}}(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha}),T)$ ***of the extreme value random process*** is called the ***reliability function*** . It represents the probability that $U_{\max}(\boldsymbol{\alpha},T)$ is equal to or less than the barrier $\tilde{b}(\boldsymbol{\alpha})=b-\boldsymbol{\mu}_U(\boldsymbol{\alpha})$ within the time interval $[0,T]$ and is commonly expressed as (Vanmarcke, 1975):

$$\boxed{L_{U_{\max}}\left(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha}),T\right)=\mathcal{P}\left[U_{\max}(\boldsymbol{\alpha},T)\le\tilde{b}(\boldsymbol{\alpha})\right]=P_{0,U}(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha}))\exp\left[-T\,\eta_U(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha}))\right],\quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right]}$$

Initial probability

$$P_{0,U}(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha}))=\mathcal{P}[U_{\max}(\boldsymbol{\alpha},0)\le\tilde{b}(\boldsymbol{\alpha})]$$

Hazard function

- By applying the classical Rice's formula:

$$\eta_U(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha}))=\nu_U^+(\boldsymbol{\alpha})\exp\left(-\frac{\tilde{b}^2(\boldsymbol{\alpha})}{2\lambda_{0,U}(\boldsymbol{\alpha})}\right)$$

Explicit Interval Spectral Moments

- By applying the classical Rice's formula and denoting with $X_{\max}(\boldsymbol{\alpha}^I, T)$ the largest dimensionless extreme process, the ***interval reliability*** can be expressed as:

dimensionless interval barrier

$$\beta_{0,U}(\boldsymbol{\alpha}^I) = \left(b - \mu_U(\boldsymbol{\alpha}^I)\right) / \sqrt{\lambda_{0,U}(\boldsymbol{\alpha}^I)}$$

$$L_{X_{\max}}\left(\boldsymbol{\alpha}, \beta_{0,X}(\boldsymbol{\alpha}), T\right) = \mathcal{P}\left[X_{\max}(\boldsymbol{\alpha}, T) \le \beta_{0,X}(\boldsymbol{\alpha})\right]$$

$$\approx \exp\left[-T\, \nu_X^+(\boldsymbol{\alpha}) \exp\left(-\frac{\beta_{0,X}^2(\boldsymbol{\alpha})}{2}\right)\right], \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = \left[\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}\right].$$

$$X_{\max}(\boldsymbol{\alpha}^I, T) = \frac{U_{\max}(\boldsymbol{\alpha}^I, T)}{\sqrt{\lambda_{0,U}(\boldsymbol{\alpha}^I)}}$$

dimensionless interval peak factor process

$$\nu_U^+(\boldsymbol{\alpha}^I) = \frac{1}{2\pi}\sqrt{\frac{\lambda_{2,U}(\boldsymbol{\alpha}^I)}{\lambda_{0,U}(\boldsymbol{\alpha}^I)}}$$

mean up-crossing rate at level $\mu_U(\boldsymbol{\alpha}^I) > 0$

Zero and second-order interval spectral moments

$$\lambda_{0,U}(\boldsymbol{\alpha}^I) \equiv \sigma_U^2(\boldsymbol{\alpha}^I) = 2\int_0^\infty G_{UU}^I(\boldsymbol{\alpha}, \omega)\,\mathrm{d}\omega; \quad \lambda_{2,U}(\boldsymbol{\alpha}^I) \equiv \sigma_{\dot{U}}^2(\boldsymbol{\alpha}^I) = =2\int_0^\infty \omega^2\, G_{UU}^I(\boldsymbol{\alpha}, \omega)\,\mathrm{d}\omega$$

Explicit Interval Reliability Sensitivity

- The ***interval reliability sensitivity*** can be derived *analytically* by differentiating $L^I_{X_{\max}}$ with respect to the deviation amplitude $\Delta\alpha_i$ of the i-*th* uncertain parameter:

$$s^I_{L_{X_{\max}},i}\left(\beta^{(0)}_{0,Y},T\right)=\left.\frac{\partial L^I_{X_{\max}}\left(\boldsymbol{\alpha},\beta_{0,U}(\boldsymbol{\alpha}),T\right)}{\partial\Delta\alpha_i}\right|_{\Delta\boldsymbol{\alpha}=\mathbf{0}}$$

$$=C(\beta^{(0)}_{0,U},T)\left\{2\beta^{(0)}_{0,U}\sqrt{\lambda^{(0)}_{0,U}}\,s^I_{\mu_U,i}+\left[\left(\beta^{(0)}_{0,U}\right)^2-1\right]s^I_{\lambda_{0,U},i}+\frac{\lambda^{(0)}_{0,U}}{\lambda^{(0)}_{2,U}}s^I_{\lambda_{2,U},i}\right\}$$

Interval sensitivity of the mean-value

Interval sensitivities of the zero -and second -order spectral moments

in terms of ***interval sensitivities of the mean-value*** $s^I_{\mu_U,i}$, ***zero-order*** $s^I_{\lambda_{0,U},i}$ and ***second -order*** $s^I_{\lambda_{2,U},i}$ ***spectral moments***.

➢ Since explicit relationships between interval statistics of the response and interval parameters have been determined, ***interval sensitivities*** can be evaluated analytically by direct differentiation with respect to the uncertain parameters.

Bounds of the interval reliability function

- Evaluation of an analytical approximation of the interval reliability function of the peak factor, along with its sensitivities, allows to estimate its bounds.
- To this aim, $L^I_{X_{\max}}\left(\beta_{0,U},T\right)$ can be approximated by applying the ***first-order interval Taylor series expansion***:

$$\boxed{\begin{aligned} L^I_{X_{\max}}\left(\beta^{(0)}_{0,U},T\right) &= L^{(0)}_{X_{\max}}\left(\beta^{(0)}_{0,U},T\right)+\sum_{i=1}^{r} s^I_{L_{X_{\max}},i}\left(\beta^{(0)}_{0,U},T\right)\Delta\alpha_i \\ &= L^{(0)}_{X_{\max}}\left(\beta^{(0)}_{0,U},T\right)+\sum_{i=1}^{r} \Delta s_{L_{X_{\max}},i}\left(\beta^{(0)}_{0,U},T\right)\Delta\alpha_i\,\hat{e}^I_i \end{aligned}}$$

deviation amplitude

nominal or midpoint CDF

with $s^I_{L_{X_{\max}},i}\left(\beta^{(0)}_{0,U},T\right)=\Delta s_{L_{X_{\max}},i}\left(\beta^{(0)}_{0,U},T\right)\hat{e}^I_i$

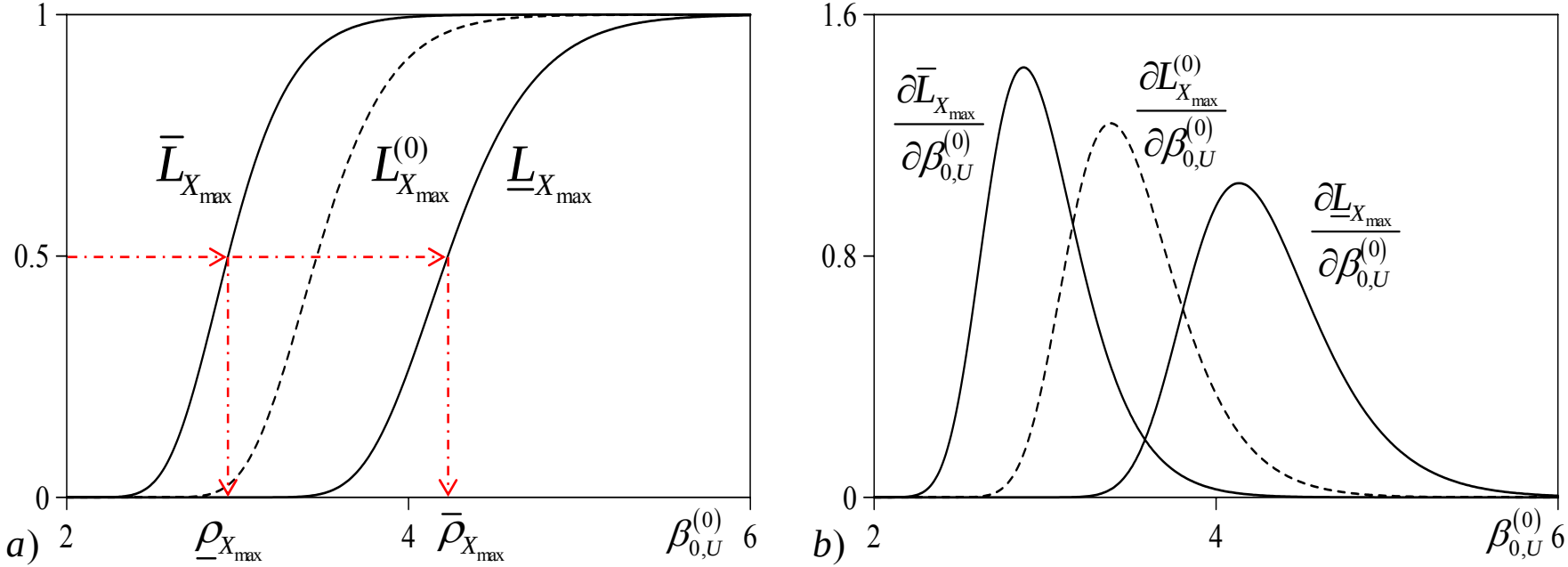
i-th peak factor reliability sensitivity

- By applying the ***improved interval analysis via EUI***, the ***Lower Bound*** and the ***Upper Bound*** of $L^I_{X_{\max}}\left(\beta_{0,U},T\right)$ can be evaluated as follows:

$$\boxed{\begin{aligned} \underline{L}_{X_{\max}}\left(\boldsymbol{\alpha},\beta^{(0)}_{0,U},T\right) &= L^{(0)}_{X_{\max}}\left(\beta^{(0)}_{0,U},T\right)-\sum_{i=1}^{r}\left|\Delta s_{L_{X_{\max}},i}\left(\beta^{(0)}_{0,U},T\right)\right|\Delta\alpha_i; \\ \overline{L}_{X_{\max}}\left(\boldsymbol{\alpha},\beta^{(0)}_{0,U},T\right) &= L^{(0)}_{X_{\max}}\left(\beta^{(0)}_{0,U},T\right)+\sum_{i=1}^{r}\left|\Delta s_{L_{X_{\max}},i}\left(\beta^{(0)}_{0,U},T\right)\right|\Delta\alpha_i. \end{aligned}}$$

Bounds of the interval reliability function

- The knowledge of the bounds of the interval reliability function of the peak factor allows to evaluate the parameters of the interval response process needed to guarantee the desired safety level.



Sketch of: a) *UB* and *LB* of the *peak factor CDF* and b) *PDFs* of the *UB* and *LB* of the *peak factor CDF*.

- $\rho_{X_{max}}(p,\boldsymbol{\alpha},T)$ denotes the ***interval fractile of the peak factor*** of order p and can be evaluated as solution of the following nonlinear interval equation:

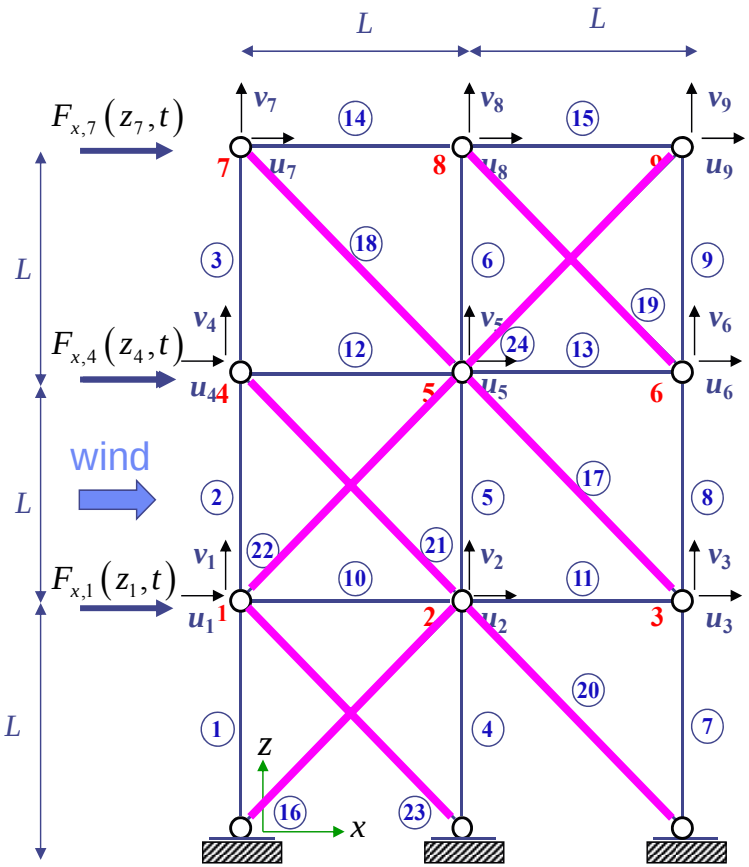
$$\boxed{p = L_{X_{max}}\left(\boldsymbol{\alpha}, \rho_{X_{max}}\left(p,\boldsymbol{\alpha},T\right),T\right)}$$

Outline

1. Improved Interval Analysis
2. Interval Stochastic Analysis
3. Explicit Interval Reliability Function
4. **Numerical Applications**
5. Concluding Remarks

Numerical Application

- 24-bar truss structure subjected to turbulent wind loads in the *x*-direction with uncertain-but-bounded Young's moduli of the diagonal bars (r=9)



- $F_{x,i}(z_i,t) = F_{x,i}^{(s)} + \tilde{F}_{x,i}(z_i,t)$

$$\approx \frac{1}{2}\rho C_D A_i w_s^2 + \rho C_D A_i \tilde{W}(z_i,t) w_s, \quad (i = 1,4,7)$$

- Wind velocity $W(z,t) = w_s(z) + \tilde{W}(z,t)$

$w_s(z) = w_{s,10}\left(z/10\right)^{\gamma}$ **Mean value**

$\tilde{W}(z,t)$ **Fluctuating component modelled as a zero-mean stationary Gaussian random field**

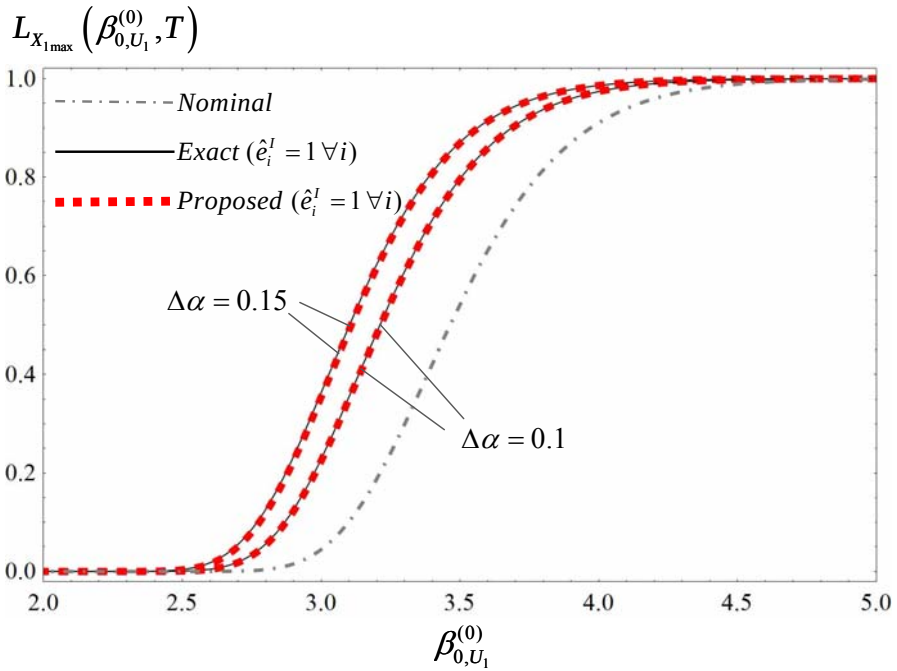
$$G_{\tilde{W}\tilde{W}}(\omega) = 4K_0 w_{s,10}^2 \frac{\chi^2}{\omega\left(1+\chi^2\right)^{4/3}}$$ **(Davenport)**

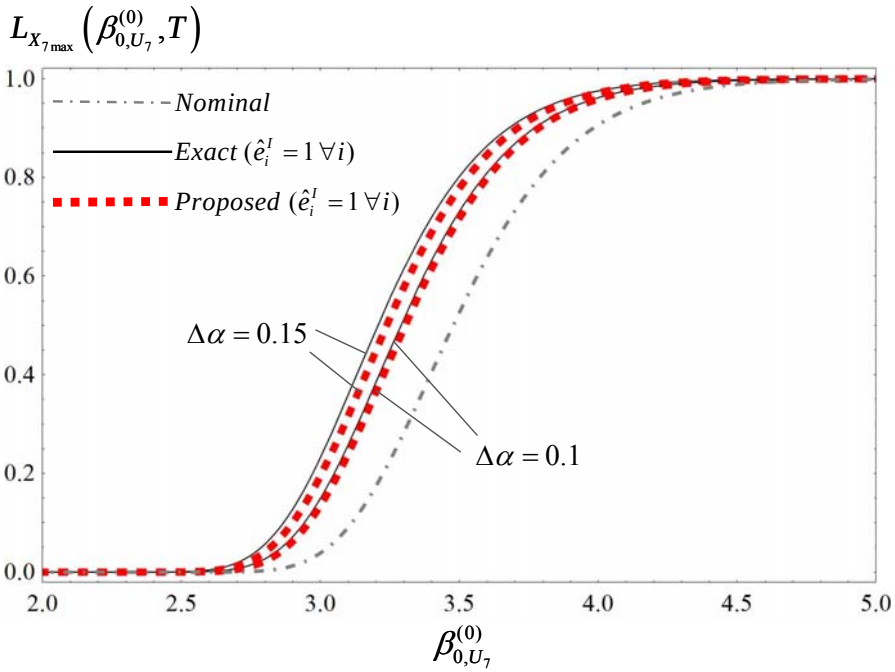
- $E_i^I = E_0\left(1+\Delta\alpha_i \hat{e}_i^I\right), (i = 16,17,\ldots,24)$ with $E_0 = 2.1\times10^8$ kN/m^2 and $\Delta\alpha_i = \Delta\alpha$

- $A_{0,i} = A_0 = 5\times10^{-4}$ m^2 $(i = 1,2,\ldots,24)$; L=3 m; M=500kg; c_0=3.517897s^{-1} and c_1=0.000547s

Numerical Results/1

- **Comparison** between the **proposed** and **exact** ***reliability functions*** of the peak factor processes $X_{1\max}(\boldsymbol{\alpha}^I, T)$ and $X_{7\max}(\boldsymbol{\alpha}^I, T)$ of the horizontal displacements of nodes 1 and 7 of the truss structure (T=1000T_0) setting the interval Young moduli of the diagonal bars at their upper bounds.

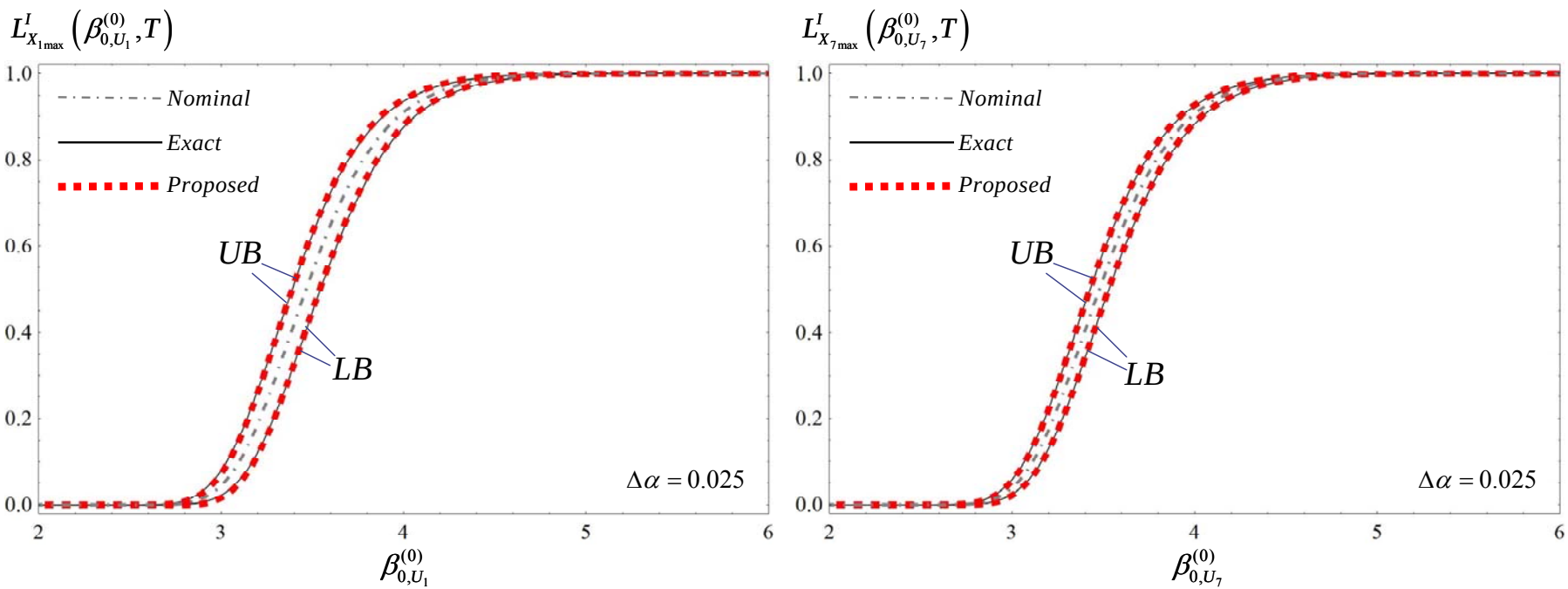




$$\overline{E}_i = E_0(1 + \Delta\alpha i) \;\; (i = 16, 17, \ldots, 24)$$

Numerical Results/2

- **Comparison** between the **exact and proposed *UB*** and ***LB*** of ***reliability functions*** of the peak factor processes $X_{1\max}(\alpha^I, T)$ and $X_{7\max}(\alpha^I, T)$ of the horizontal displacements of nodes 1 and 7 of the truss structure with interval Young's moduli of the diagonal bars.

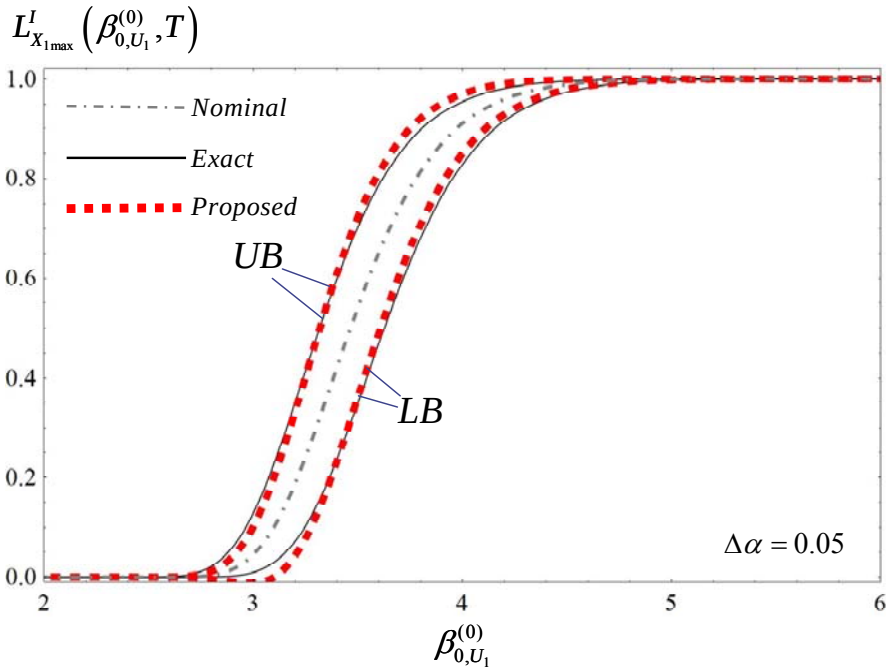


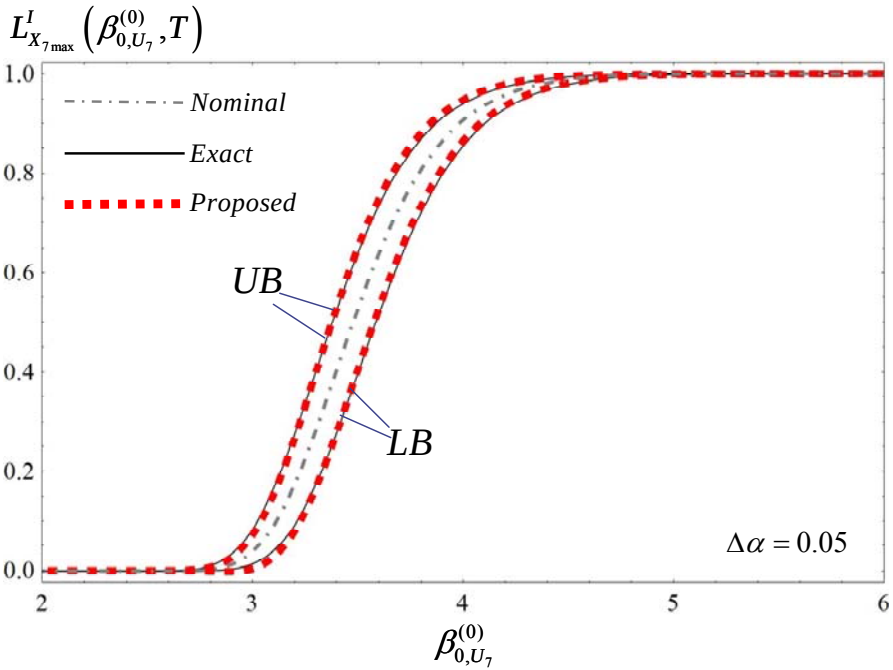
$$E_i^I = E_0\left(1 + \Delta\alpha_i \hat{e}_i^I\right), (i = 16, 17, \ldots, 24)$$

$$\boxed{\Delta\alpha_i = \Delta\alpha = 0.025} \; ; \quad T = 1000 T_0$$

Numerical Results/3

- **Comparison** between the **exact and proposed *UB*** and ***LB*** of ***reliability functions*** of the peak factor processes $X_{1\max}(\alpha^I, T)$ and $X_{7\max}(\alpha^I, T)$ of the horizontal displacements of nodes 1 and 7 of the truss structure with interval Young's moduli of the diagonal bars.





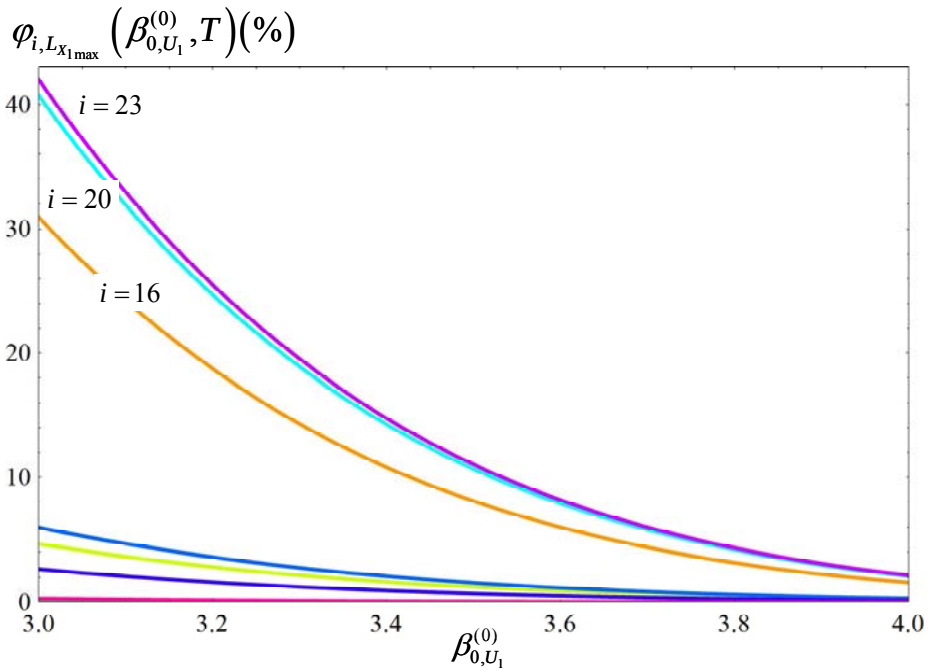
$$E_i^I = E_0\left(1 + \Delta\alpha_i \hat{e}_i^I\right), (i = 16, 17, \ldots, 24)$$

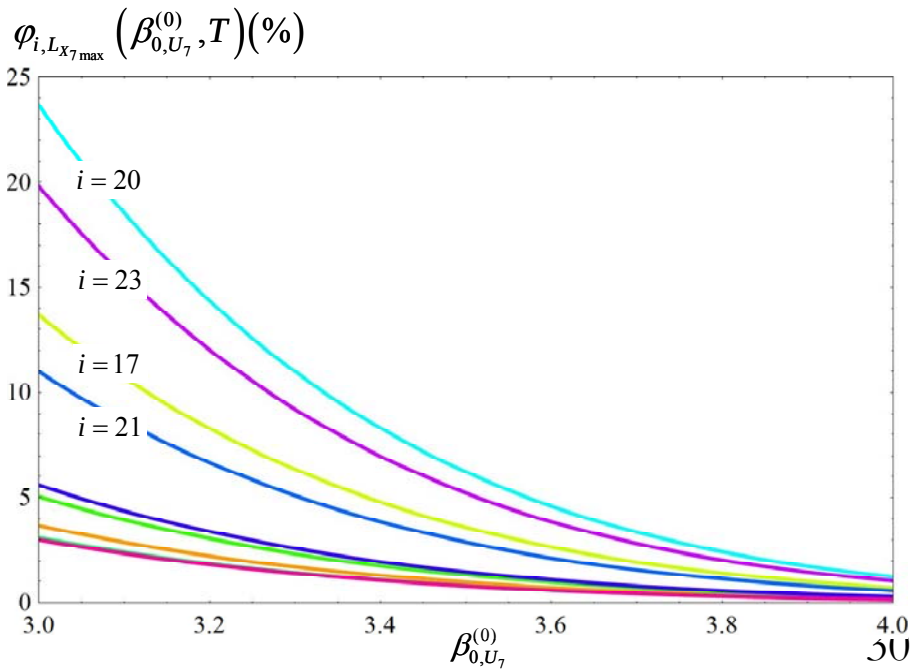
$$\boxed{\Delta\alpha_i = \Delta\alpha = 0.05}\ ; \quad T = 1000T_0$$

Numerical Results/4: sensitivity analysis

- The proposed approximate closed-form expression of the ***peak factor interval reliability sensitivity*** $s^{I}_{L_{X_{\max}},i}\left(\beta^{(0)}_{0,U},T\right)$ is applied to investigate the rate of change in the interval *CDF* due to changes in the structural parameters. To identify the most influential uncertain parameters, a percentage measure of the influence of the generic interval variable on the *CDF* of the selected peak factor process can be defined by introducing a ***function of sensitivity***:

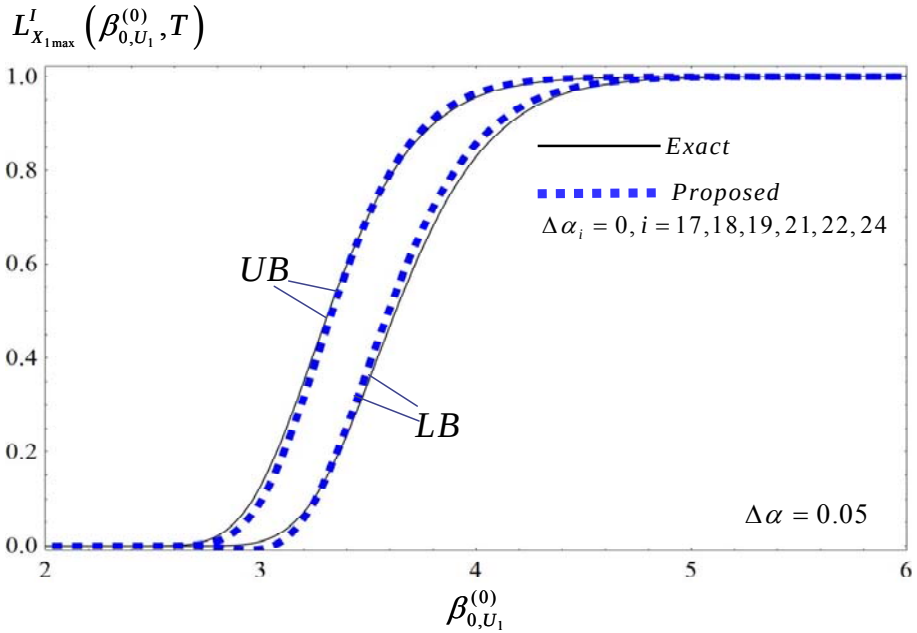
$$\boxed{\varphi_{i,L_{X_{\max}}}\left(\beta^{(0)}_{0,Y},T\right)(\%)=\frac{\left|\Delta s_{L_{X_{\max}},i}\left(\beta^{(0)}_{0,Y},T\right)\right|}{L^{(0)}_{X_{\max}}\left(\beta^{(0)}_{0,Y},T\right)}\Delta\alpha_i\times 100}$$



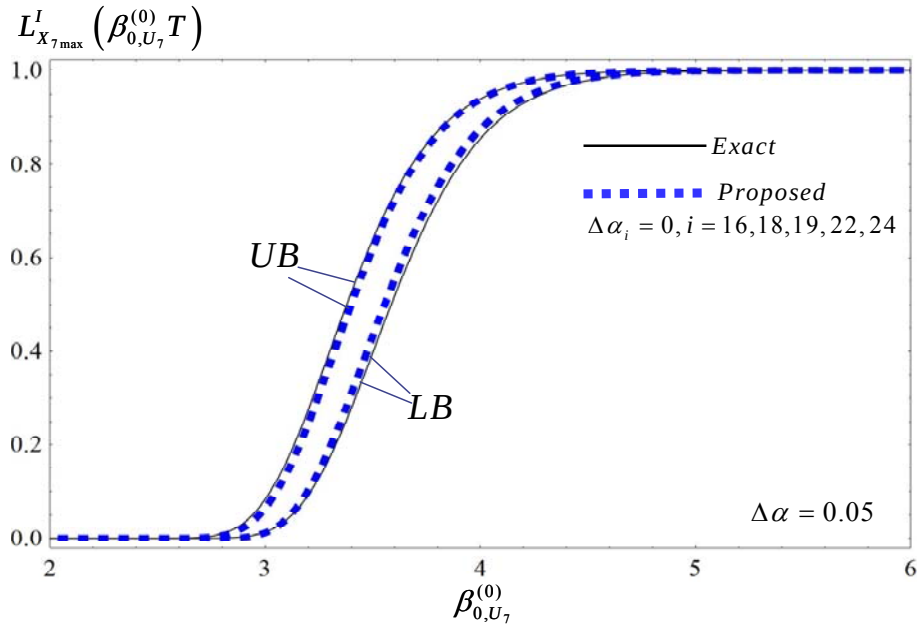


Numerical Results/5

- **Comparison** between the **exact** and **proposed** ***UB*** and ***LB*** of ***reliability functions*** of the peak factor processes $X_{1\max}(\alpha^I, T)$ and $X_{7\max}(\alpha^I, T)$ of the horizontal displacements of nodes 1 and 7 of the truss structure with interval Young's moduli of the diagonal bars, neglecting terms associated to the least influential parameters in the first-order Taylor series expansion based on the results of the **reliability sensitivity analysis**.



$E_i^I = E_0\left(1 + \Delta\alpha_i \hat{e}_i^I\right), (i = 16, 20, 23)$



$E_i^I = E_0\left(1 + \Delta\alpha_i \hat{e}_i^I\right), (i = 16, 17, 20, 21)$

$\Delta\alpha_i = \Delta\alpha = 0.05$; $T = 1000T_0$

Outline

1. Improved Interval Analysis
2. Interval Stochastic Analysis
3. Explicit Interval Reliability Function
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Conclusions

- An analytical approach to evaluate the reliability function for structures with uncertain-but-bounded parameters subjected to stationary Gaussian random excitation has been proposed.
- The interval reliability function has been evaluated in approximate closed-form by applying the ***Interval Rational Series Expansion*** in conjunction with the ***improved interval analysis***, recently developed by the authors.
- The ***Interval Rational Series Expansion*** provides an approximate explicit expression of the inverse of an interval matrix with modifications. The ***improved interval analysis*** allows to limit the overestimation of the interval solution width due to the dependency phenomenon occurring in classical interval analysis.
- Remarkable features of the proposed approach are: *i*) the capability of handling a large number of uncertainties and evaluate analytically the interval reliability function; *ii*) the possibility of providing very accurate explicit estimates of the bounds of the interval reliability in the framework of the ***first-order interval Taylor series expansion***.
- A wind-excited truss structure with interval axial stiffness of the diagonal bars has been analyzed. Appropriate comparisons with the exact reliability bounds obtained by the vertex method have demonstrated the accuracy of the proposed procedure.