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Non-Linear Analysis of Beams with Large Deflections – An Interval Finite Element Approach

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Introduction

Large Deformation Beam

Interval Finite Element Formulation.

Non-linear Equation Solving

Results

Conclusion

Structural Engineering

is the ART of molding material we ***do not understand*** into shapes **we *can not analyze*** to withstand forces we will ***not be able to measure*** in such a way that the public does not really suspect.

E. H. Brown

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Interval Finite Element Methods

- Intervals an elegant way of introducing uncertainty into finite element analysis

- Underlying technology for fuzzy analysis

- Underlying technology for P-box analysis

Interval Finite Elements

- Linear Problems
- Non-linear materials (elastic plastic and cubic)
- Non-linear geometric behavior

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Large Deformation Strain

- The kinematic equation of Euler-Bernoulli beam theory under the assumption that the beam is loaded in *x-y* plane of symmetry, the axial strain for large deformation (Bauchau and Craig, 2009) is expressed as follows:

$$\varepsilon_x(x,y) = \frac{\partial U}{\partial x} - y\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\left(\left(\frac{\partial U}{\partial x} - y\frac{\partial^2 V}{\partial x^2}\right)^2 + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2\right)$$

Partitioning of strain

Assuming $\left(\frac{\partial U}{\partial x} - y\frac{\partial^2 V}{\partial x^2}\right)^2$ to be small, the expression for $\varepsilon_x(x, y)$ is simplified as

$$\left.\begin{aligned}
\varepsilon_x(x,y) &= \varepsilon_1(x,y) + \varepsilon_2(x,y) \\
\varepsilon_1(x,y) &= \frac{\partial U}{\partial x} - y\frac{\partial^2 V}{\partial x^2} \\
\varepsilon_2(x,y) &= \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2
\end{aligned}\right\}$$

Total Potential Energy

$$\Pi = \frac{1}{2}\int_{x=0}^{L}\int_{A}\left(\varepsilon_1(x,y)+\varepsilon_2(x,y)\right)E\left(\varepsilon_1(x,y)+\varepsilon_2(x,y)\right)dA\,dx - \int_{L_1} q\,dx$$

$$\left.\begin{aligned}\Pi = \frac{1}{2}\int_{x=0}^{L}\int_{A}\varepsilon_1(x,y)E(x,y)\varepsilon_1(x,y)dA\,dx + \int_{x=0}^{L}\int_{A}\varepsilon_1(x,y)E(x,y)\varepsilon_2(x,y)dA\,dx + \\ \frac{1}{2}\int_{x=0}^{L}\int_{A}\varepsilon_2(x,y)E(x,y)\varepsilon_2(x,y)dA\,dx - \int_{L_1} q\,dx\end{aligned}\right\}$$

Introduction

Large Deformation Beam

Interval Finite Element Formulation.

Non-linear Equation Solving

Results

Conclusion

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Variational Form

$$\Pi = \Pi_1 + \Pi_2 + \Pi_3 - \int_{L_1} q\,dx$$

$$\begin{aligned}\delta\Pi = {} & \{\delta d_i\}^T \int_{x=0}^{L} \int_A B_1^T E(x,y) B_1\, dA\, dx \{d_j\} + \\ & \frac{3}{4}\{\delta d_i\}^T \int_{x=0}^{L} \int_A B_1^T E(x,y) B_2\, dA\, dx \{d_j\} + \frac{3}{4}\{\delta d_i\}^T \int_{x=0}^{L} \int_A B_2^T E(x,y) B_1\, dA\, dx \{d_j\} + \\ & \frac{1}{2}\{\delta d_i\}^T \int_{x=0}^{L} \int_A B_2^T E(x,y) B_2\, dA\, dx \{d_j\} = 0\end{aligned}$$

$$\left.\begin{aligned} \varepsilon_1(x,y) &= [B_1]\{d\} \\ \varepsilon_2(x,y) &= \frac{1}{2}\{d\}^T[B_2]\{d\} \end{aligned}\right\} \tag{9}$$

where

$$\left.\begin{aligned} [B_1] &= \left[-\frac{1}{L} \quad \frac{-(12x-6L)}{L^3}y \quad \frac{-(6xL-4L^2)}{L^3}y \quad \frac{1}{L} \quad \frac{-(-12x+6L)}{L^3}y \quad -\frac{(6xL-2L^2)}{L^3}y \right] \\ [B_2] &= [N']^T[N'] \end{aligned}\right\}$$

$$[N_1'] = \left[0 \quad \frac{(6x^2-6xL)}{L^3} \quad \frac{(3x^2L-4xL^2+L^3)}{L^3} \quad 0 \quad \frac{(-6x^2+6xL)}{L^3} \quad \frac{(3x^2L-2xL^2)}{L^3} \right]$$

$$[K_1] = \int_{x=0}^{L} \int_{A} B_1^T E(x,y) B_1 dA dx$$

$$[K_2] = \frac{3}{4} AE \int_{x=0}^{L} \left[B_1^T B_2 + B_2^T B_1 \right] dx$$

$$[K_3] = \frac{1}{2} AE \int_{x=0}^{L} B_2^T B_2 dx$$

Interval Stiffness Matrix

$$\left.\begin{aligned}
[K_1] &= \int_{x=0}^{L}\int_A [B_1]^T E[B_1]\,dA\,dx \\
[K_2] &= \frac{3}{4}A\left(\int_{x=0}^{L}[B_{1mid}]^T \boldsymbol{\eta} E\left[N'\right]dx + \int_{x=0}^{L}\left[N'\right]^T \boldsymbol{\eta} E[B_{1mid}]\,dx\right) \\
[K_3] &= \frac{1}{2}A\int_{x=0}^{L}\left[N'\right]^T \boldsymbol{\eta}^2 E\left[N'\right]dx
\end{aligned}\right\}$$

$$[K] = \begin{bmatrix} \\ A_1^T & A_1^T & A_2^T & A_2^T \\ \\ \\ \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \eta & & \\ & & \eta & \\ & & & \eta^2 \end{bmatrix} \begin{bmatrix} D_1 & & & \\ & D_{1mid} & & \\ & & D_{1mid} & \\ & & & D_{1mid} \end{bmatrix} \begin{bmatrix} A_1 & & & \\ A_2 & & & \\ A_1 & & & \\ A_2 & & & \end{bmatrix}$$

$$\begin{bmatrix} [G_1][\eta][D][G_2] & C^T \\ C & 0 \end{bmatrix} \begin{Bmatrix} U \\ \lambda \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

$$(K + B\,\mathbf{D}\,A)\mathbf{u} = a + F\,\mathbf{b}$$

$$C := (K + BD_0 A)^{-1}. \qquad \textbf{(9)}$$

where D_0 is chosen to ensure invertability (often D_0 is selected as the midpoint of **D**), the solution **u** can be written as:

$$\mathbf{u} = (Ca) + (CF)\mathbf{b} + (CB)\mathbf{d}. \qquad \textbf{(10)}$$

To obtain a solution with tight interval enclosure we define two auxiliary interval quantities,

$$\begin{aligned} \mathbf{v} &= A\mathbf{u} \\ \mathbf{d} &= (D_0 - \mathbf{D})\mathbf{v} \end{aligned} . \qquad \textbf{(11)}$$

which, given an initial estimate for **u**, we iterate as follows:

$$\mathbf{v} = \{(ACa) + (ACF)\mathbf{b} + (ACB)\mathbf{d}\} \cap \mathbf{v}, \quad \mathbf{d} = \{(D_0 - \mathbf{D})\mathbf{v}\} \cap \mathbf{d}. \qquad \textbf{(12)}$$

Error in secondary quantities

Conventional Finite Element

$$\Pi = \frac{1}{2} U^T K U - U^T P$$

with the conditions

$$\frac{\partial \Pi}{\partial U_i} = 0 \qquad \textbf{\textit{for all}}\ i$$

Secondary quantities such as stress/strain calculated from displacement have shown significant overestimation of interval bounds

Use constraints to augment original Variational

$$\Pi^* = \frac{1}{2}U^T K U - U^T P + \lambda_1^T (C_1 U - V) + \lambda_2^T (C_2 U - \mathcal{E})$$

$$\begin{pmatrix} K & C_1^T & C_2^T & 0 \\ C_1 & 0 & 0 & 0 \\ C_2 & 0 & 0 & -I \\ 0 & 0 & -I & 0 \end{pmatrix} \begin{pmatrix} U \\ \lambda_1 \\ \lambda_2 \\ \mathcal{E} \end{pmatrix} = \begin{pmatrix} P \\ V \\ 0 \\ 0 \end{pmatrix}$$

Introduction

Large Deformation Beam

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Non-linear Equation Solving

Results

Conclusion

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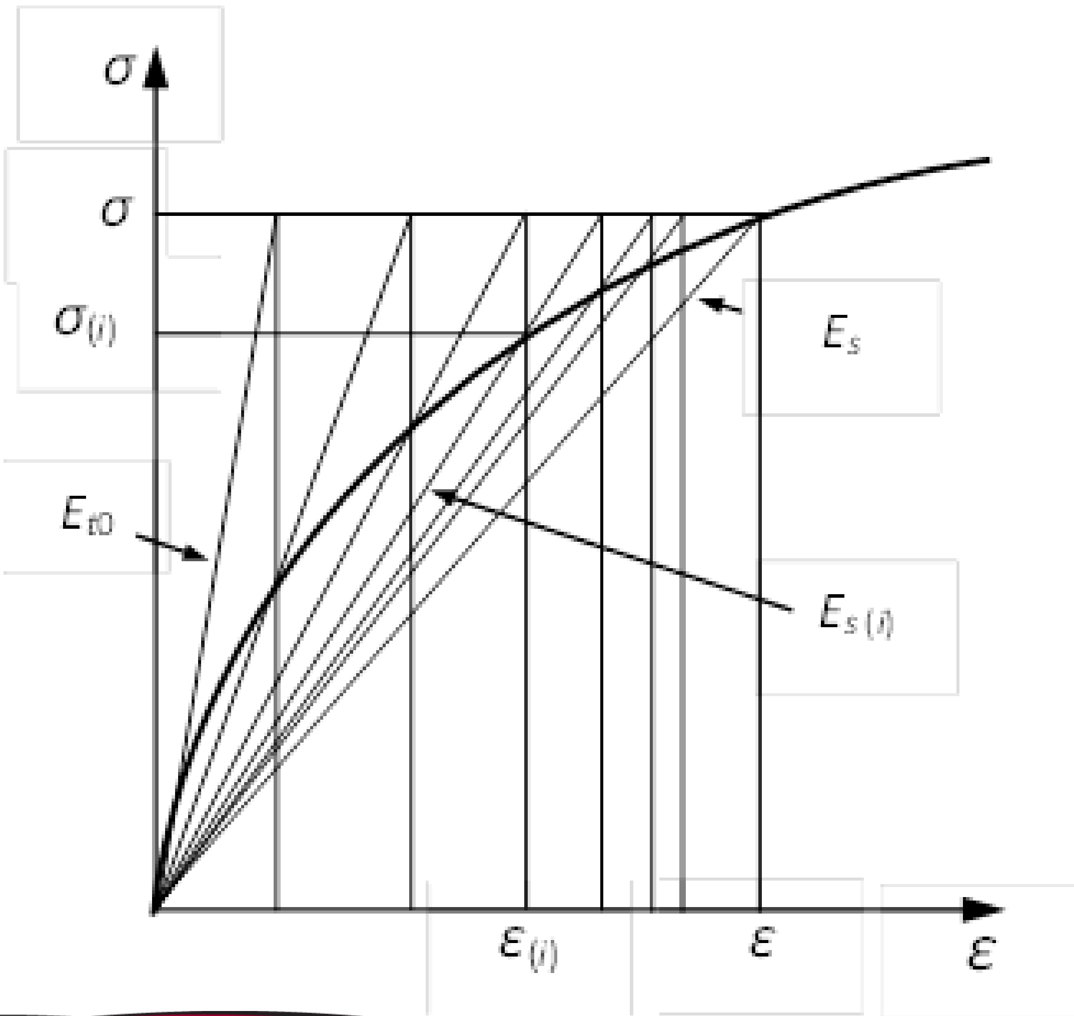
Non-linear equation strategies

- Secant methods
- Newton like methods

Secant Method



Interval Modified Newton-Raphson Method

$$K_t \mathbf{U} = \mathbf{P},$$

$$K_t \mathbf{U} = M\mathbf{d},$$

$$\mathbf{F}_i = \boldsymbol{\sigma}_i A_i = (a\boldsymbol{\varepsilon}_i + b\boldsymbol{\varepsilon}_i^3)A_i,$$

$$\mathbf{F}_i = \{a\begin{pmatrix}\varepsilon_{i1} & \varepsilon_{i2}\end{pmatrix}\begin{pmatrix}\mathbf{d}_1 \\ \mathbf{d}_2\end{pmatrix} + b\,\mathbf{[}\begin{pmatrix}\varepsilon_{i1} & \varepsilon_{i2}\end{pmatrix}\begin{pmatrix}\mathbf{d}_1 \\ \mathbf{d}_2\end{pmatrix}\mathbf{]}^3\}A_i,$$

$$\mathbf{F} = \begin{pmatrix} \mathbf{F}_{1x} \\ \mathbf{F}_{1y} \\ \vdots \\ \mathbf{F}_{mx} \\ \mathbf{F}_{my} \end{pmatrix} = \begin{pmatrix} c_{1x1} & c_{1x2} \\ c_{1y1} & c_{1y2} \\ \vdots & \vdots \\ c_{mx1} & c_{mx2} \\ c_{my1} & c_{my2} \end{pmatrix} \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix} + \begin{pmatrix} c_{1x3} & c_{1x4} \\ c_{1y1} & c_{1y2} \\ \vdots & \vdots \\ c_{mx3} & c_{mx4} \\ c_{my3} & c_{my4} \end{pmatrix} \begin{pmatrix} \mathbf{d}_1^3 \\ \mathbf{d}_2^3 \end{pmatrix} + \begin{pmatrix} c_{1x5} & c_{1x6} \\ c_{1y5} & c_{1y6} \\ \vdots & \vdots \\ c_{mx5} & c_{mx6} \\ c_{my5} & c_{my6} \end{pmatrix} \begin{pmatrix} \mathbf{d}_1\mathbf{d}_2^2 \\ \mathbf{d}_1^2\mathbf{d}_2 \end{pmatrix}$$

The '*out of balance*' force vector can now be introduced as

$$\delta\mathbf{F} = (M + MM_1)\begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{pmatrix} + MM_2\begin{pmatrix} \mathbf{d}_1^3 \\ \mathbf{d}_2^3 \end{pmatrix} + MM_3\begin{pmatrix} \mathbf{d}_1\mathbf{d}_2^2 \\ \mathbf{d}_1^2\mathbf{d}_2 \end{pmatrix}$$

Containment as a stopping criterion

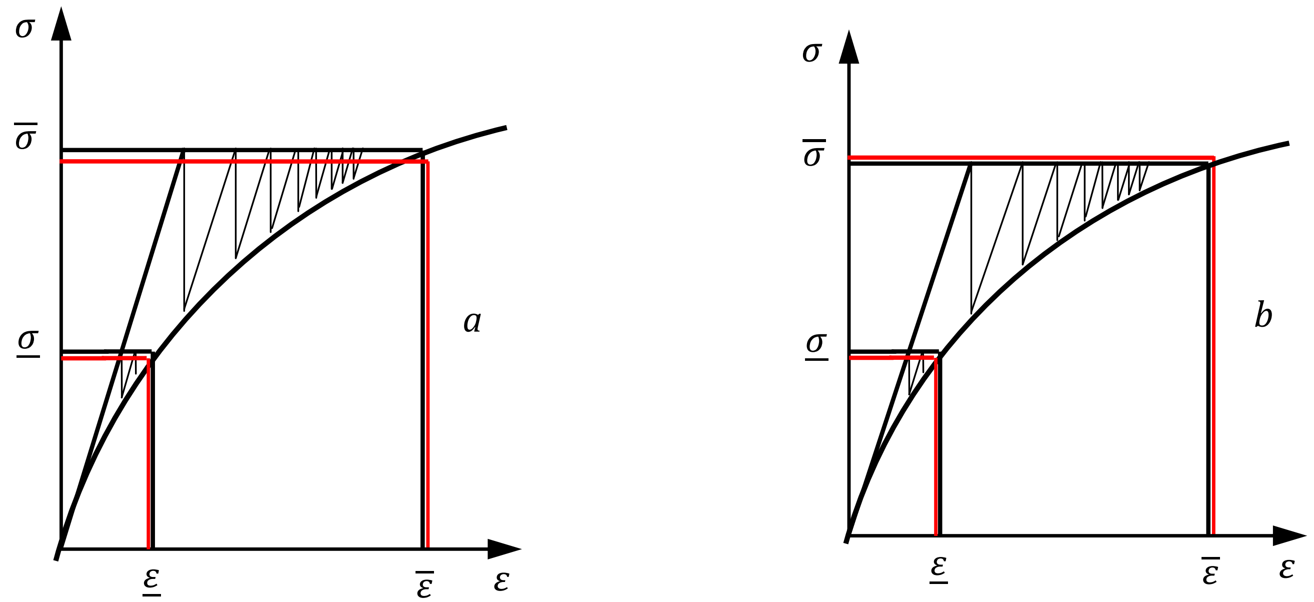


Figure 2. Stress-strain relationship, Modified Newton-Raphson method. *a*) before containment, *b*) after containment

Introduction

Large Deformation Beam

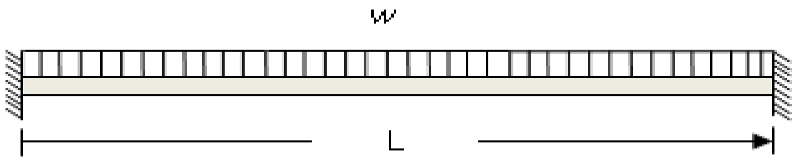
Interval Finite Element Formulation.

Non-linear Equation Solving

Results

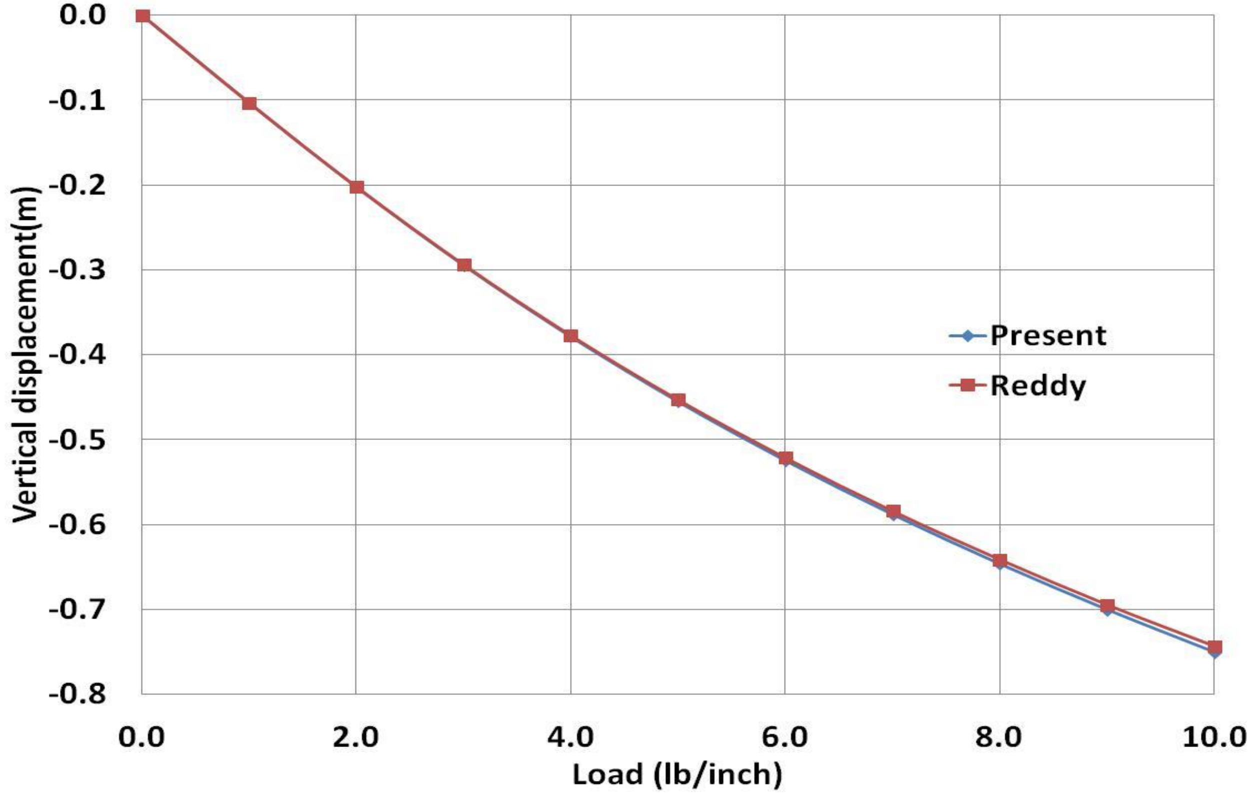
Conclusion

Fixed-Fixed Beam

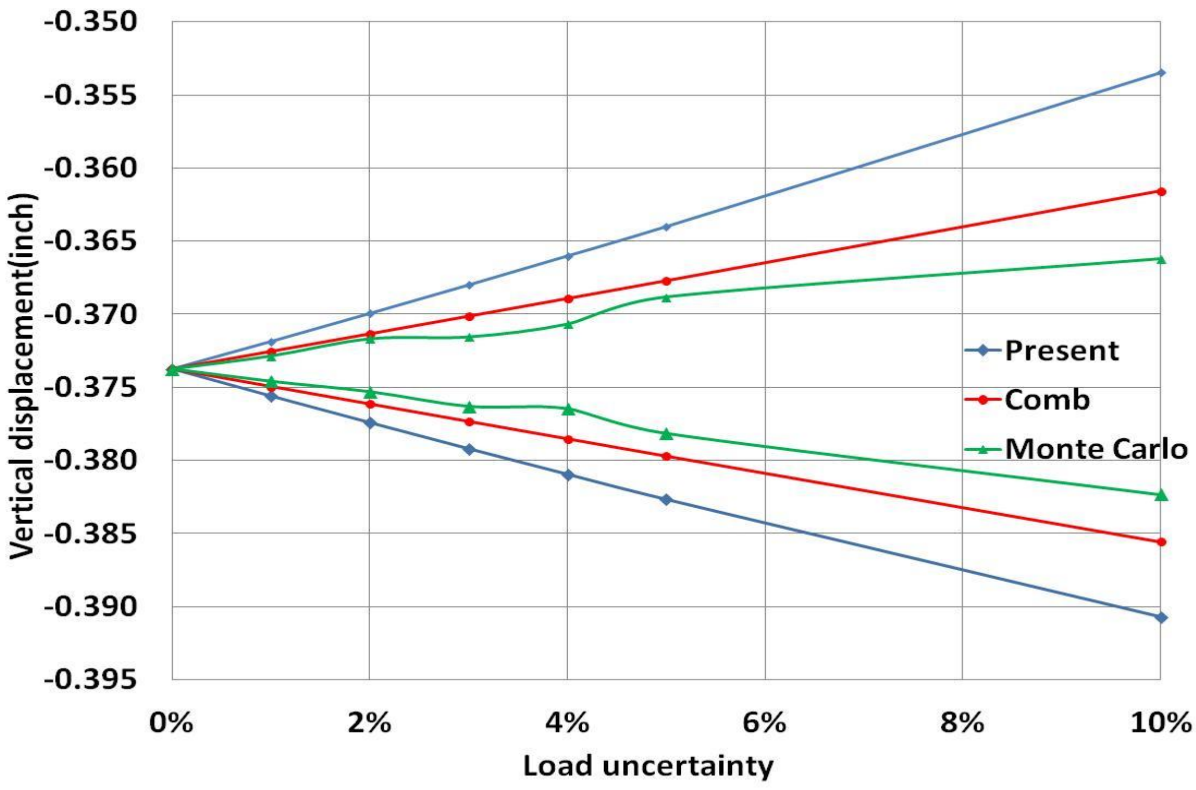


L= 100 in (2.54 m)
w=10 lb/in. (1.751kN/m)
Cross section 1 in x 1in.
30ksi (207 Gpa)

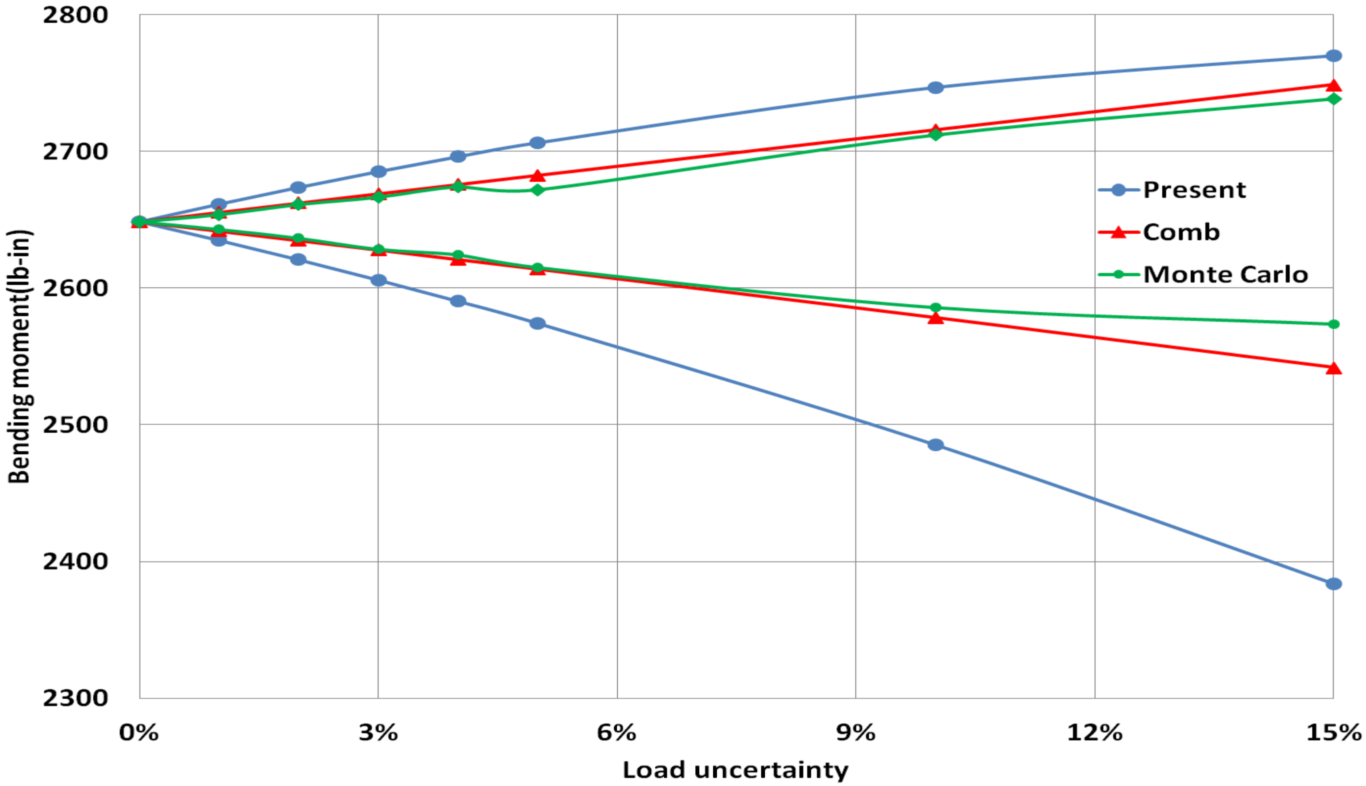
Fixed-Fixed Beam
Load-Deflection Behavior



Fixed-Fixed Beam Center Deflection



Bending Moment Fixed-Fixed Beam



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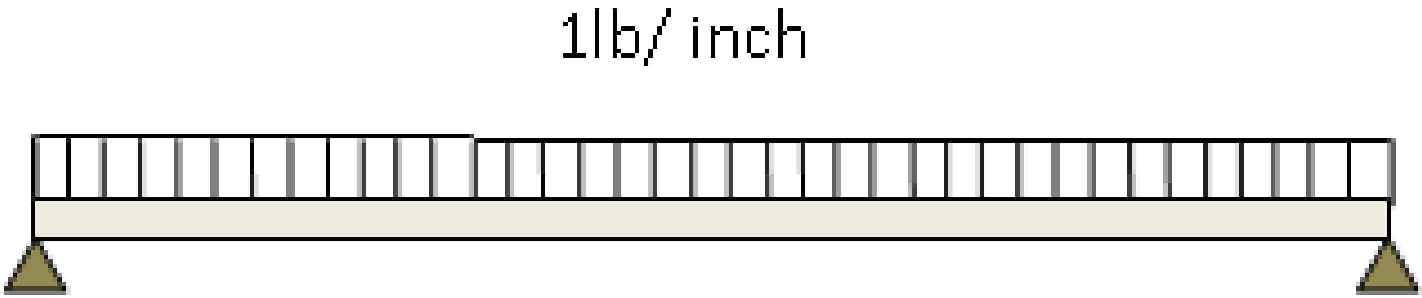
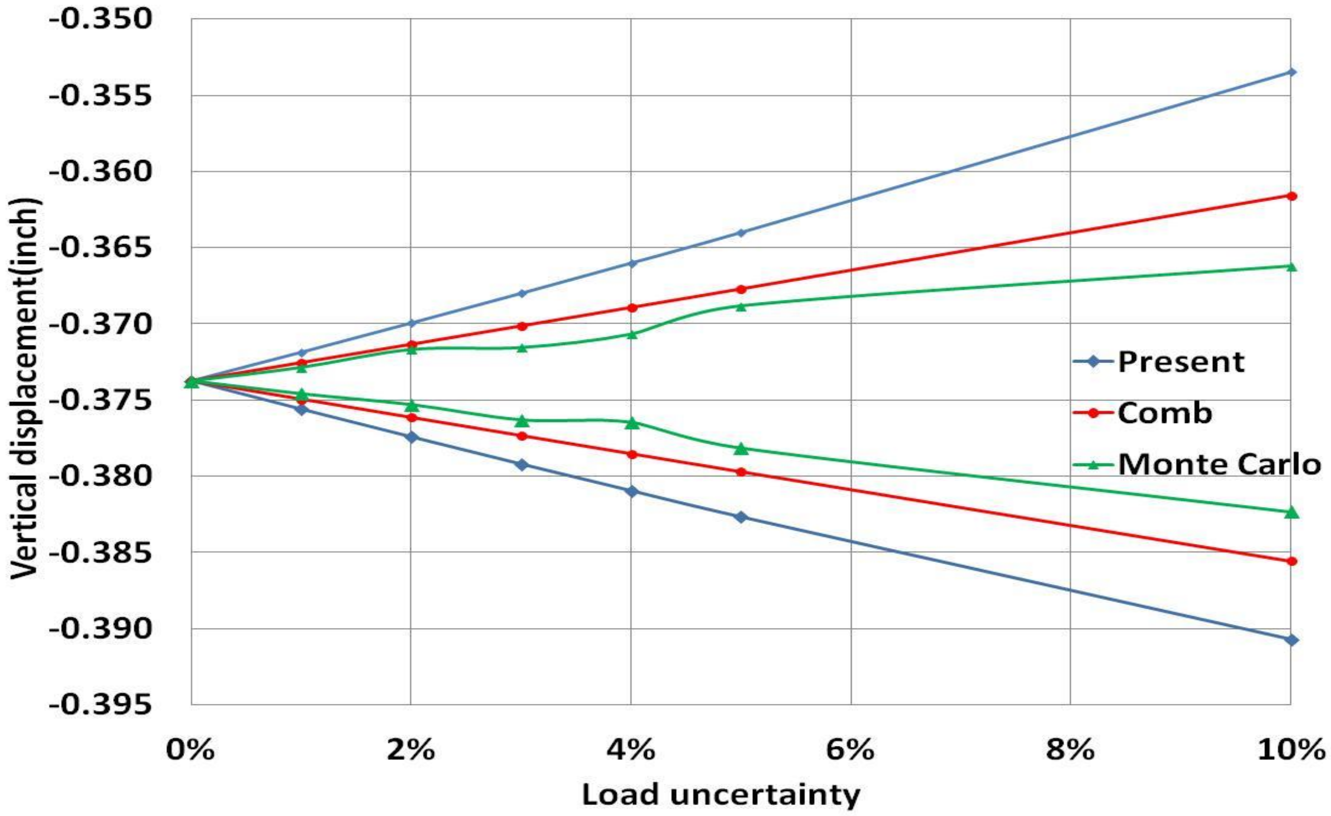


L= 100 in (2.54 m)
w=10 lb/in. (1.751kN/m)
Cross section 1 in x 1in.
30ksi (207 Gpa)

Vertical Displacement in Beam



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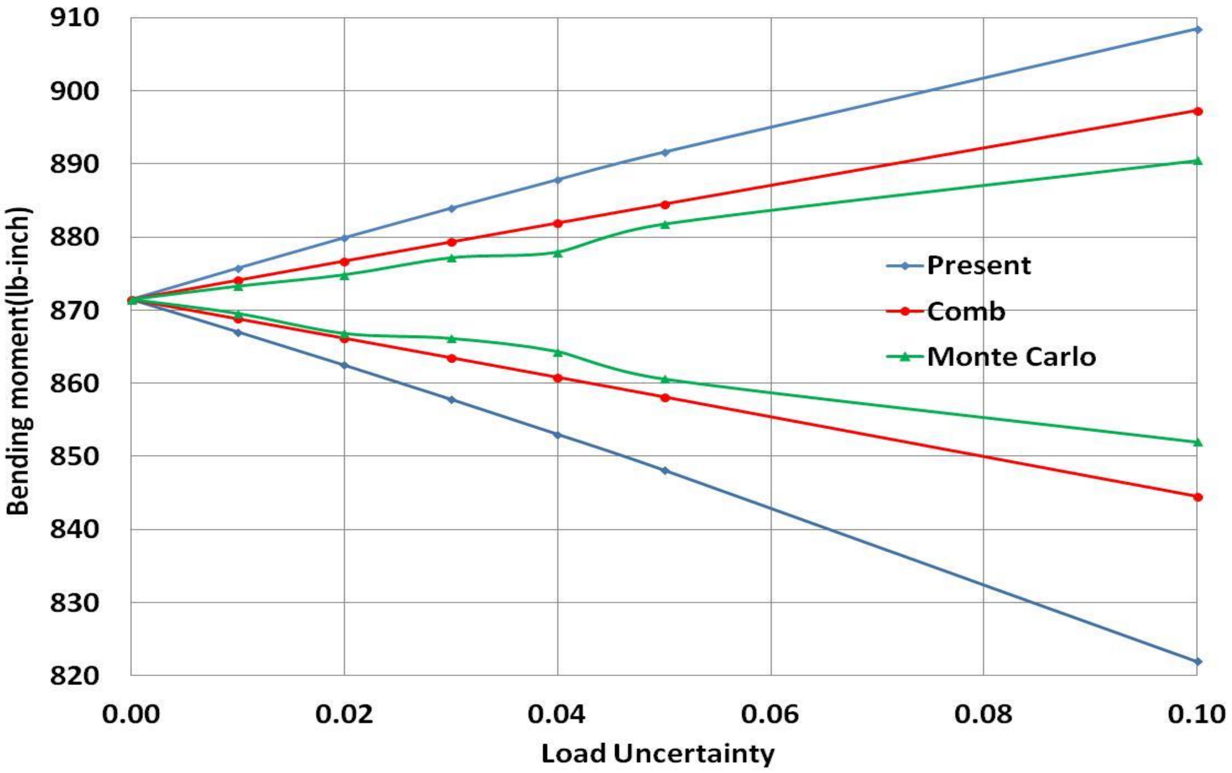
Bending Moment in Beam



Introduction

Large Deformation Beam

Interval Finite Element Formulation.

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Results

Conclusion

Summary

- Nonlinear analysis of structures with Large Deformation can be performed using interval analysis with reasonably sharp bounds on response
- Multiple occurrences of nonlinear interval terms leaves room for additional improvement
- Formulation for combined material and geometric non-linear behavior is straight forward.

1801

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Rene Magritte, Clairvoyance, 1936