Recent Advances in Reliability Estimation of Time-Dependent Problems Using the Concept of Composite Limit State

Zissimos P. Mourelatos
Monica Majcher
Vijitashwa Pandey

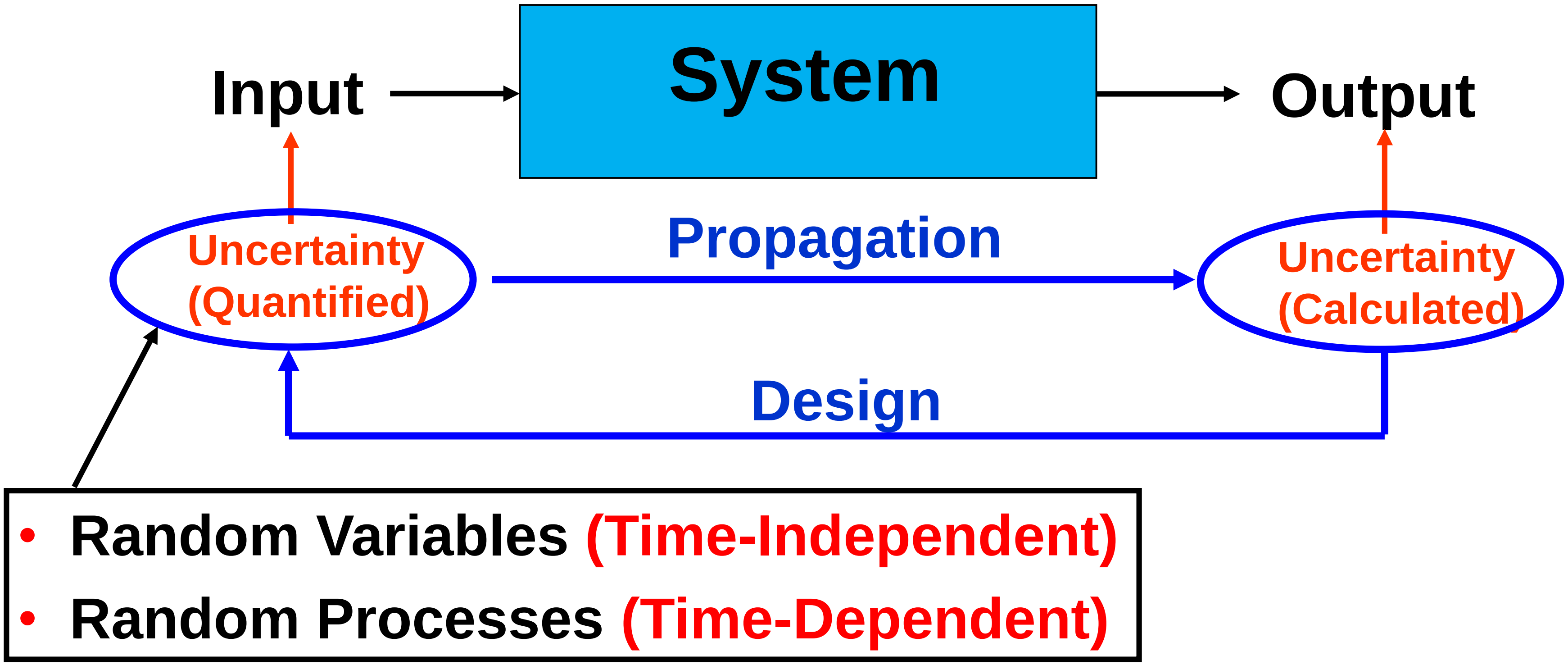
Mechanical Engineering Department
Oakland University

REC 2014; Chicago, IL

Overview

- Background information
  - Definition of time-dependent P_f
  - Out-crossing rate approach
- Proposed approach to estimate time-dependent P_f using Composite Limit State (CLS)
  - Identification of CLS
  - Calculation of time-dependent P_f
- Implementation points
- Summary and future work

Background



Input

System

Output

Uncertainty (Quantified)

Propagation

Uncertainty (Calculated)

Design

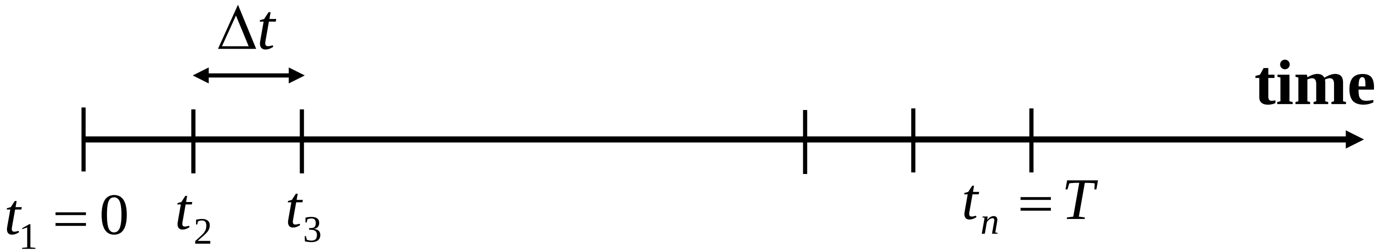
- **Random Variables (Time-Independent)**
- **Random Processes (Time-Dependent)**

Challenges:

- **Quantification of a Random Processes**
- **Estimation of time-dependent reliability**

Time-Dependent Probability of Failure

$$P_f\left(0,T\right) = P\left\{\exists t \in \left[0,T\right] : g\left(\mathbf{X}, \mathbf{Y}\left(t\right), t\right) \le 0 \right\}$$



$$P_f\left(0,T\right) = P\left(\bigcup_{i=1}^{n} g\left(\mathbf{X}, \mathbf{Y}\left(t_i\right), t_i\right) \le 0 \right)$$

Series System Reliability Problem

Time-Dependent Probability of Failure

$$P_f(0,T) = 1 - \left(1 - P_f^i(0)\right)\exp\left\{-\int_0^T \lambda(t)\,\mathrm{d}t\right\}$$

Failure rate

Out-crossing rate approach

$$\lambda(t) \approx \nu^+(t)$$

Up-crossing rate

$$\nu^+(t) = \lim_{\Delta\tau \to 0} \frac{P\left[g\left(\mathbf{X}, \mathbf{Y}(t), t\right) > 0 \cap g\left(\mathbf{X}, \mathbf{Y}(t+\Delta\tau), t+\Delta\tau\right) \le 0\right]}{\Delta\tau}$$

Accurate ONLY if up-crossings are statistically independent

Characterization of Input Random Process

K-L expansion is used to represent random process *Y(t)* as :

$$\boxed{Y(t)=\mu_Y(t)+\sum_{i=1}^{p}\sqrt{\lambda_i}\cdot\mathbf{\Phi}_i^T(t)\cdot Z_i}$$

$\mu_Y(t)$: mean of random process *Y(t)*
$\lambda, \mathbf{\Phi}(t)$: eigenvalue and eigenvector of covariance matrix of *Y(t)*
Z : standard normal random variable

$\boxed{Y(t) \text{ is assumed Gaussian}}$

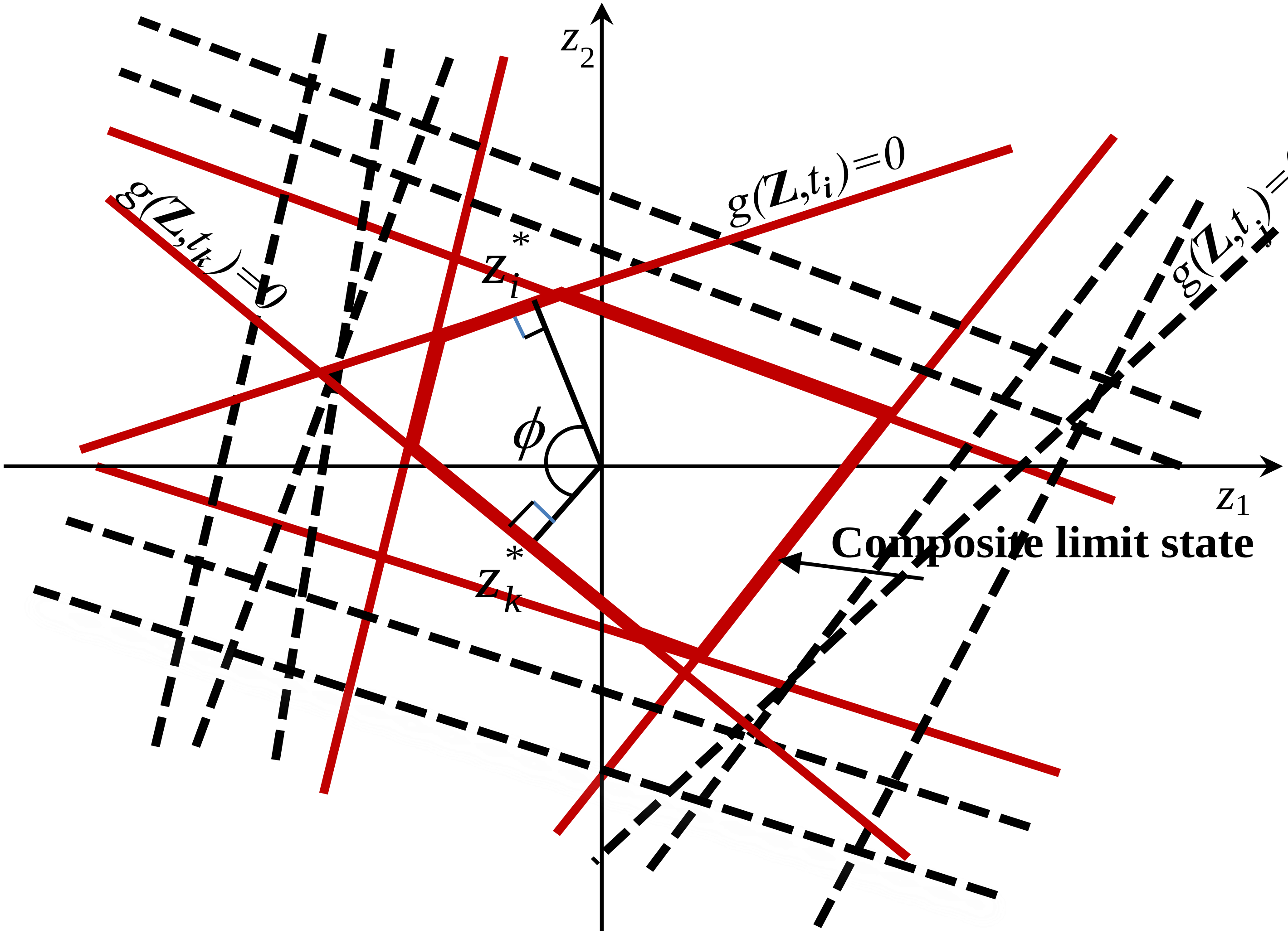
Calculation of Time-Dependent Probability of Failure (P_f)

$$P_f\left(0,T\right)=P\left\{\exists t\in\left[0,T\right]:g\left(\mathbf{X},\mathbf{Y}\left(t\right),t\right)\le 0\right\}$$

Concept of Composite Limit State is used

The Composite Limit State defines a convex domain representing the intersection of safe regions of all instantaneous limit states

Identification of Composite Limit State

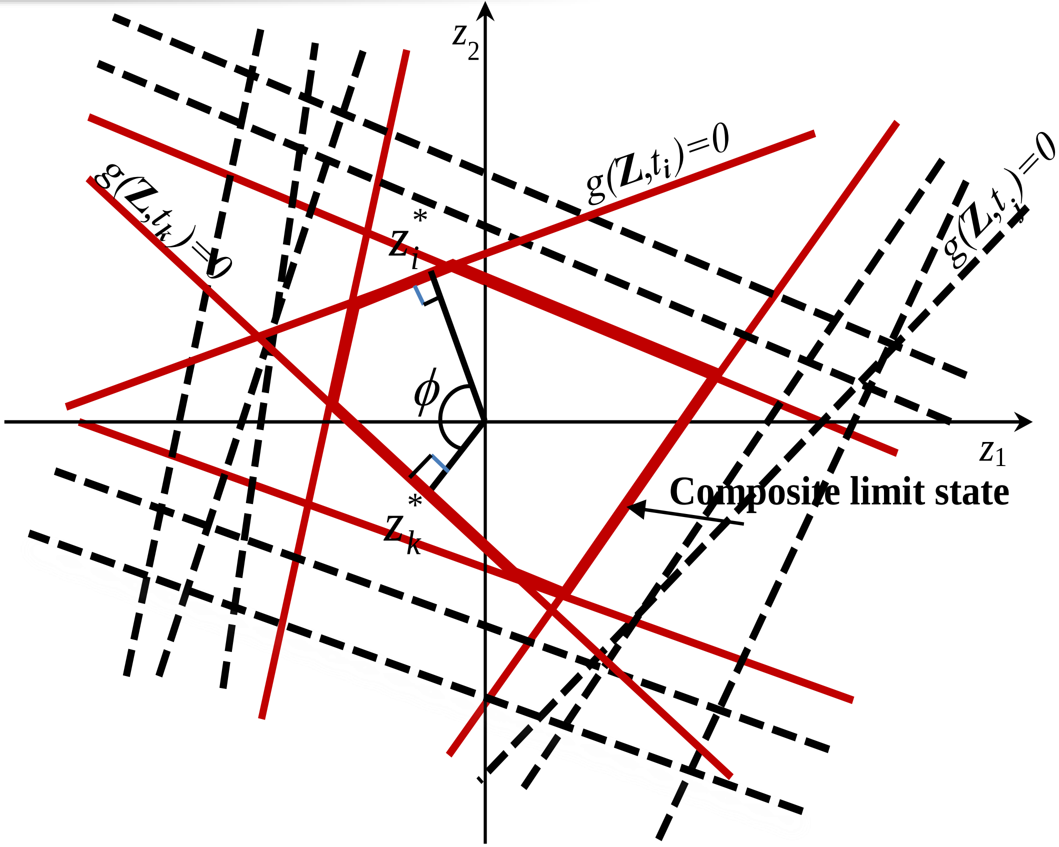


$$\boxed{\rho_{ik} = \cos(\phi)}$$

Correlation Coefficient

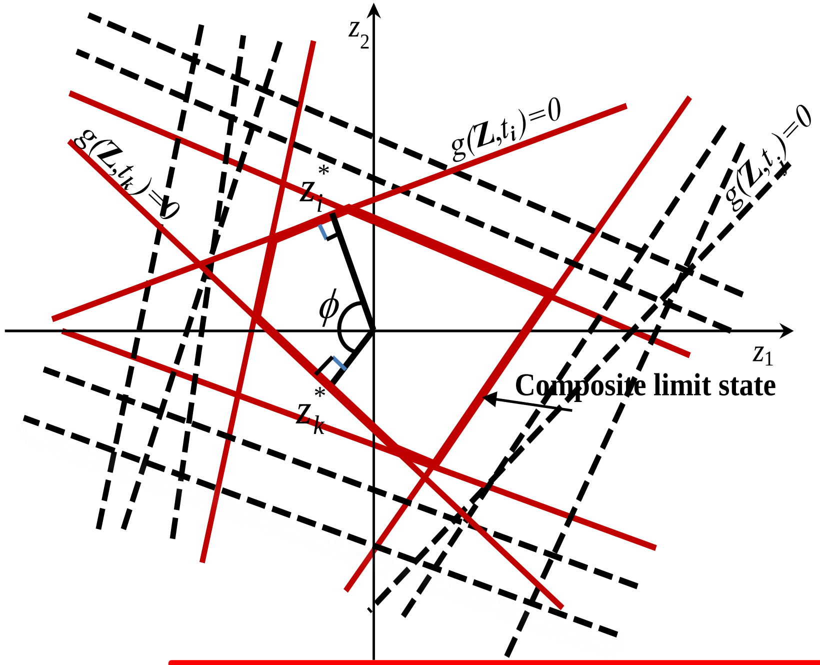
Two-step approach to identify composite limit state

Identification of Composite Limit State



Step 1: **Delete highly correlated** $\left(\rho_{ij} \geq +0.99\right)$ **instantaneous limit states (almost parallel)**

Identification of Composite Limit State



Step 2:

Delete instantaneous limit states that are not part of the composite

If set $\left\{ \mathbf{Z} : g\left(\mathbf{Z}, t_j\right) = 0 \,, \bigcap_{i=1, i \neq j} g\left(\mathbf{Z}, t_i\right) \geq 0 \right\}$ **is null,**

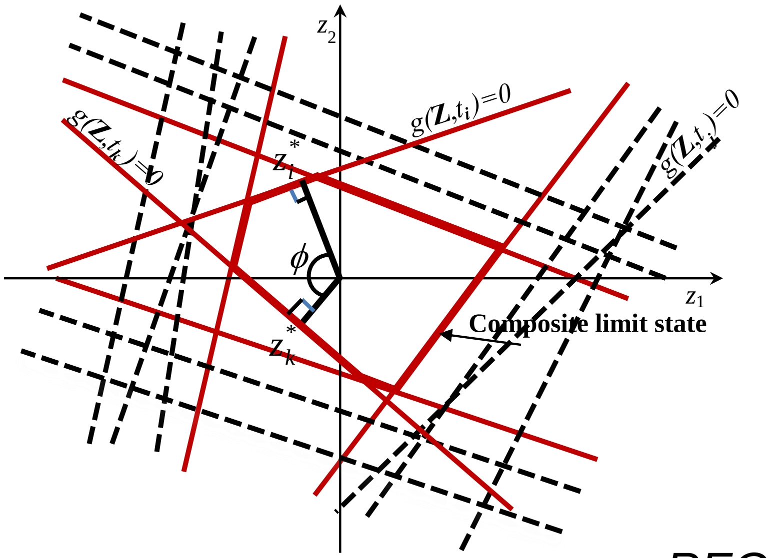
the j^{th} limit state is deleted

Check by solving a series of LPs

Calculation of Time-Dependent P_f

$$\Phi\left(-\beta_i\right)$$

$$P\left(\cup_{i=1}^{l} E_i\right) = \sum_{i \le l} P\left(E_i\right) - \sum_{i_1 < i_2} P\left(E_{i_1} \cap E_{i_2}\right) + \sum_{i_1 < i_2 < i_3} P\left(E_{i_1} \cap E_{i_2} \cap E_{i_3}\right) - \cdots + (-1)^{l} \sum_{i_1 < i_2 \ldots < i_{l-1}} P\left(E_{i_1} \cap \ldots \cap E_{i_{l-1}}\right) + (-1)^{l+1} P\left(E_1 \cap E_2 \cap \ldots \cap E_l\right)$$



$$P_{ij}^{f} = \Phi\left(-\beta\left(t_i\right)\right)\Phi\left(-\beta\left(t_j\right)\right) + \int_{0}^{\rho_{ij}} \varphi\left(-\beta\left(t_i\right), -\beta\left(t_j\right); z\right) dz$$

Bivariate standard normal vector

Calculation of Time-Dependent P_f

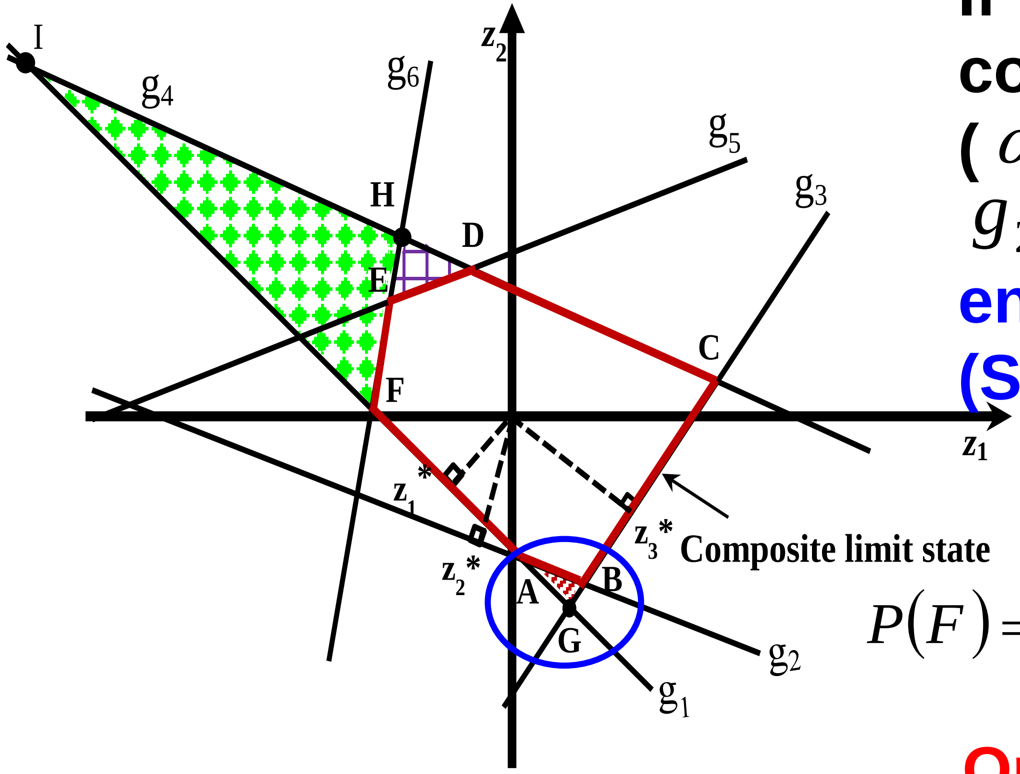
Our approach calculates P_f exactly as

$$P\left(\bigcup_{i=1}^{l} E_i\right) = \sum_{i\le l} P(E_i) - \sum_{i_1<i_2} P\left(E_{i_1}\cap E_{i_2}\right)$$

by eliminating ALL other terms using the convex polyhedron of the safe domain

This is a substantial contribution in both Time-Dependent Reliability and System Reliability

Calculation of Time-Dependent P_f



If $\mathbf{z}_2^*$ is a positive linear combination of $\mathbf{z}_1^*$ and $\mathbf{z}_3^*$, ($\alpha_j \mathbf{z}_j^* + \alpha_k \mathbf{z}_k^* = \mathbf{z}_i^*$, $\alpha_j, \alpha_k > 0$) g_2 can be eliminated enlarging the safe domain (SD) so that :

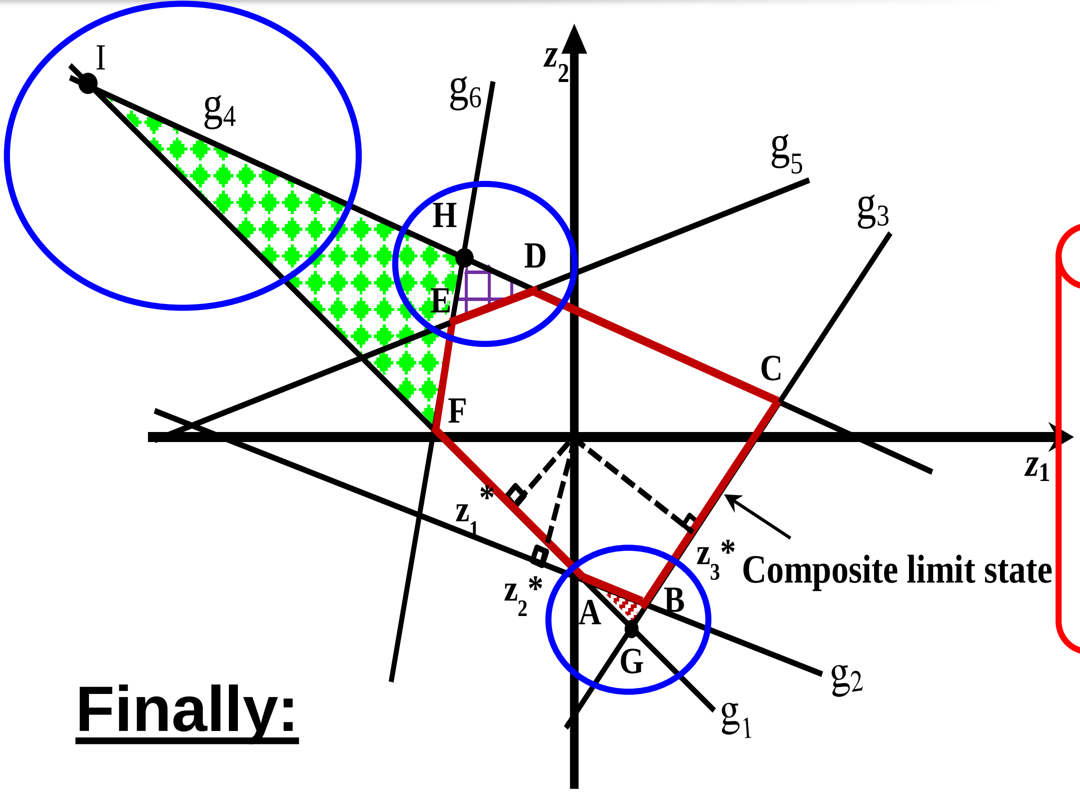
$$P(F) = 1 - P_{ABCDEFA} = \left(1 - P_{GCDEFG}\right) + P^f_{ABG}$$

Original SD

Enlarged SD

$$P^f_{ABG} = P\left(g_1 \ge 0,\ g_2 \le 0,\ g_3 \ge 0\right) = \Phi\left(-\beta_2\right) - P^f_{12} - P^f_{23} + P^f_{13}$$

Calculation of Time-Dependent P_f



This is an **EXACT** calculation of P_f involving the convex polyhedron of the SD

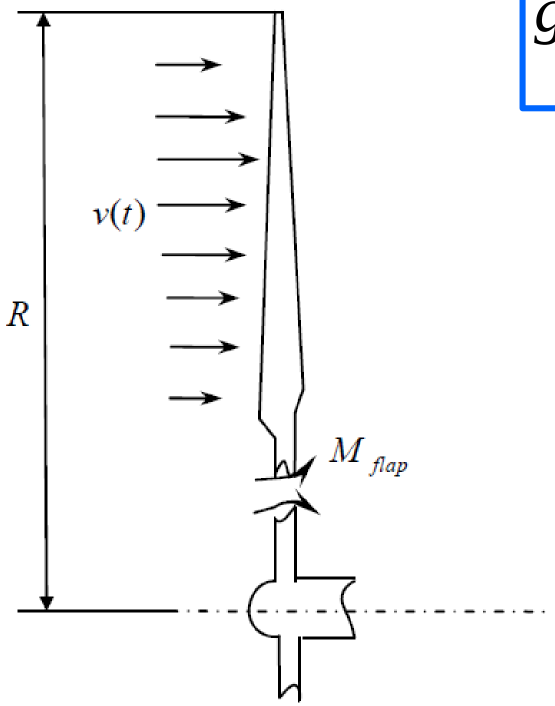
Finally:

$$P(F) = 1 - P_{ABCDEFA} = \left(1 - P_{GCDEFG}\right) + P^f_{ABG} = \left(1 - P_{GCHFG}\right) + P^f_{ABG} + P^f_{DHE} = \left(1 - P_{GCI}\right) + P^f_{ABG} + P^f_{DHE} + P^f_{FHI}$$

$$1 - P_{GCI} = \Phi\left(-\beta_3\right) + \Phi\left(-\beta_4\right) + \Phi\left(-\beta_1\right) - P^f_{34} - P^f_{41} - P^f_{13}$$

Illustration of Composite Limit State

Example : Hydrokinetic Turbine Blade under Time-dependent River Flow Loading*



$$\boxed{g(t)=\varepsilon_{allow}-\frac{M_{flap}t_1}{EI}=\varepsilon_{allow}-\frac{\rho\cdot v^2(t)\cdot C_m\cdot t_1}{2EI}}$$

ε_{allow} **: Allowable strain**

$\varepsilon_{allow}, t_1, I$ **: Random variables**

ρ, C_m, E **: Constants**

$v(t)$ **: Gaussian random process with autocorrelation coefficient function**

$$\rho_v(t_1,t_2)=\cos(2\pi(t_2-t_1))$$

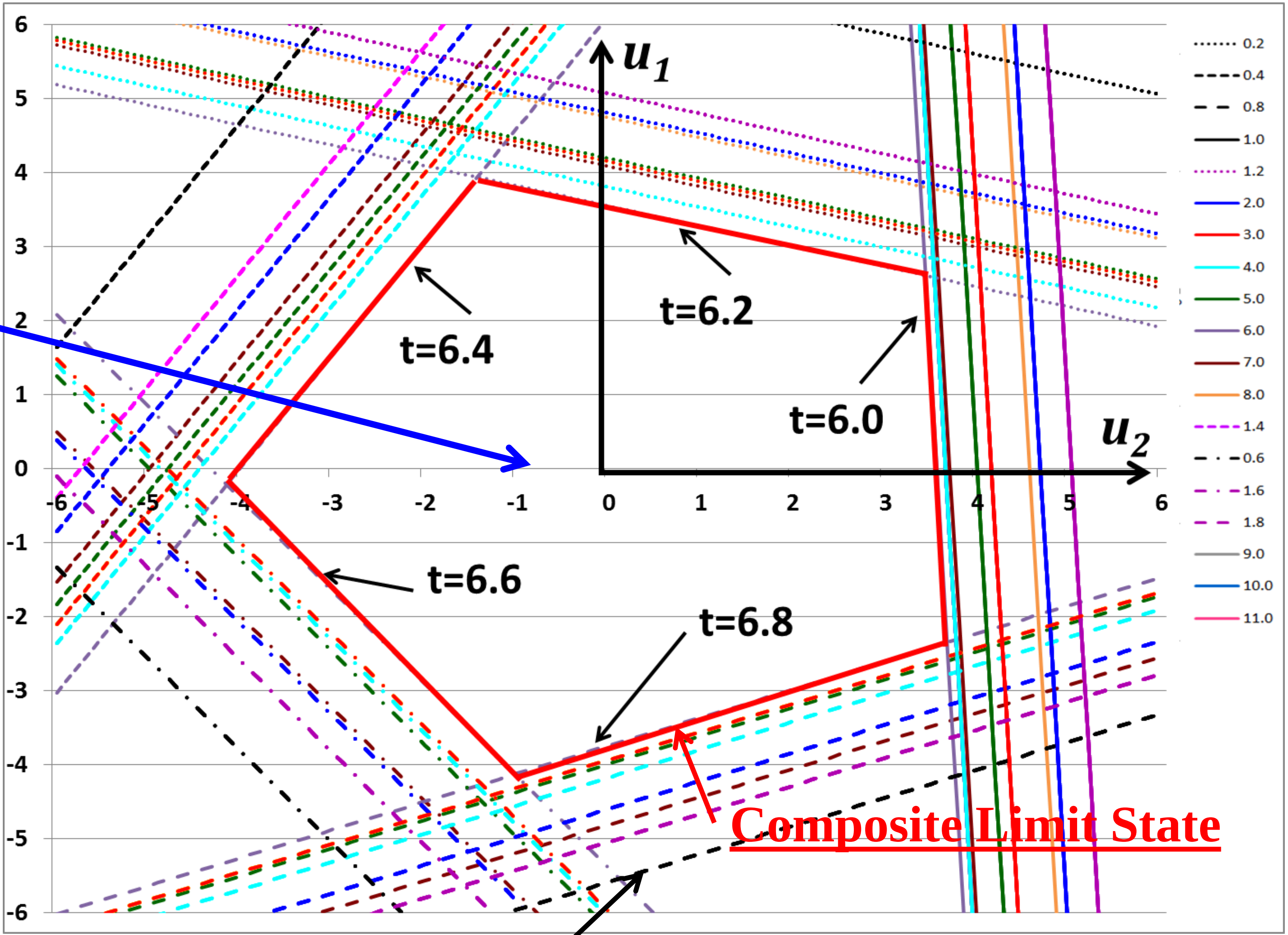
P_F **is calculated from 0 to 12 months**

*Hu, Z. and Du, X., (2012), "Time-dependent Reliability Analysis by a Sampling Approach to Extreme Values of Stochastic Processes," *Proceedings of the ASME 2012 IDETC/CIE*

Illustration of Composite Limit State

t = 8.2 months

Safe Region



Composite Limit State

Instantaneous Limit State

Illustration of Composite Limit State

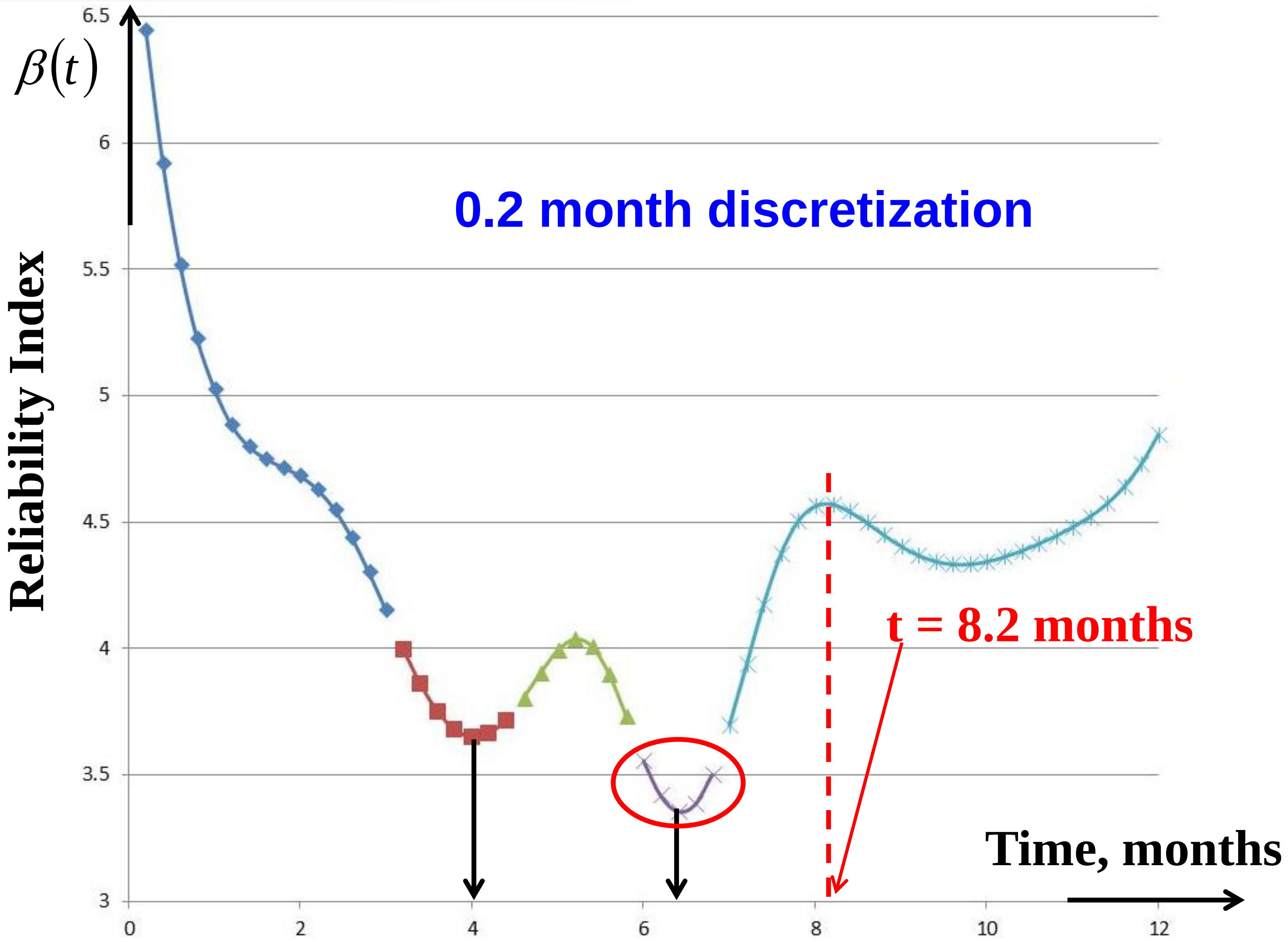
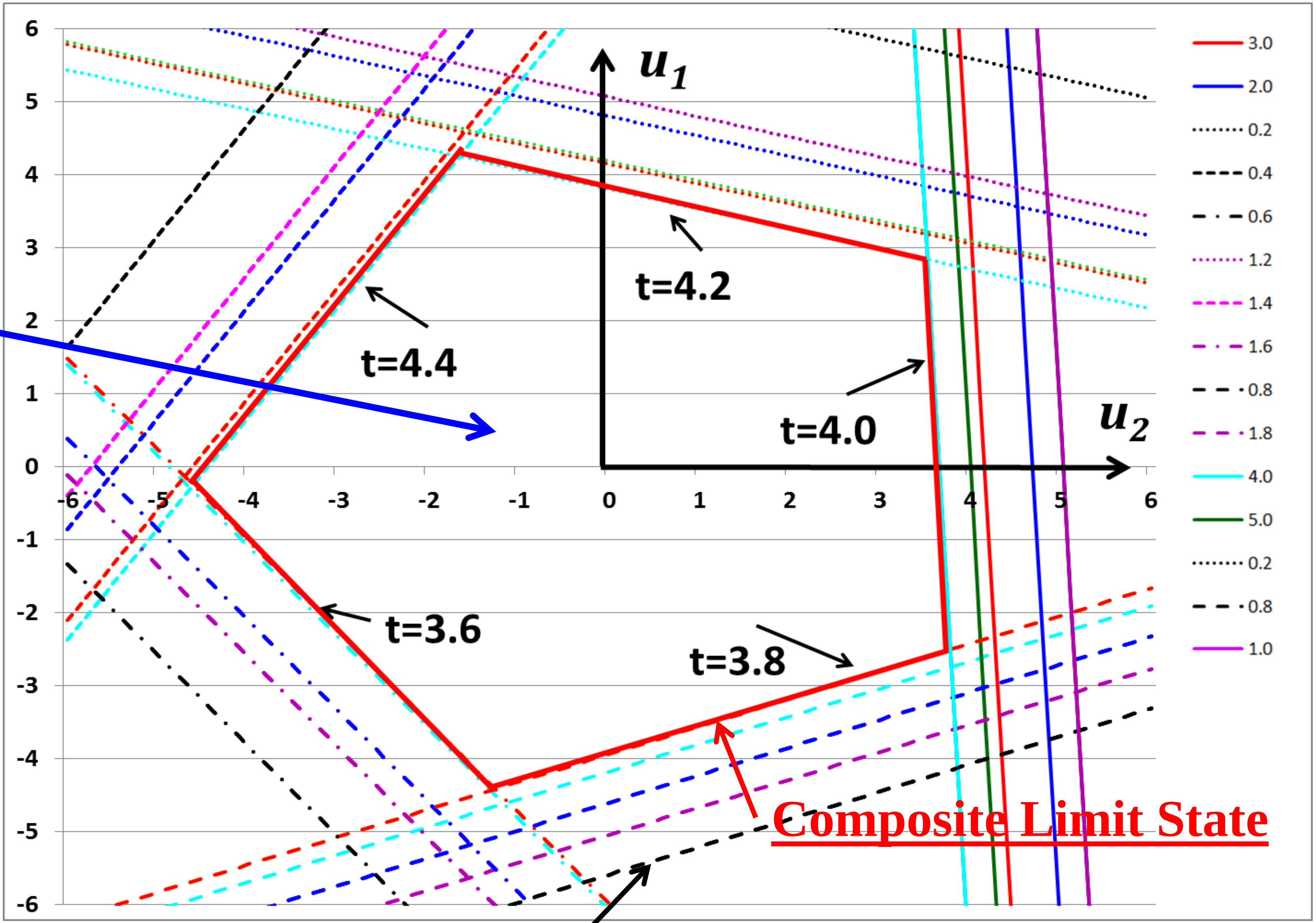


Illustration of Composite Limit State

t = 5.2 months

Safe Region



Composite Limit State

Instantaneous Limit State

Illustration of Composite Limit State

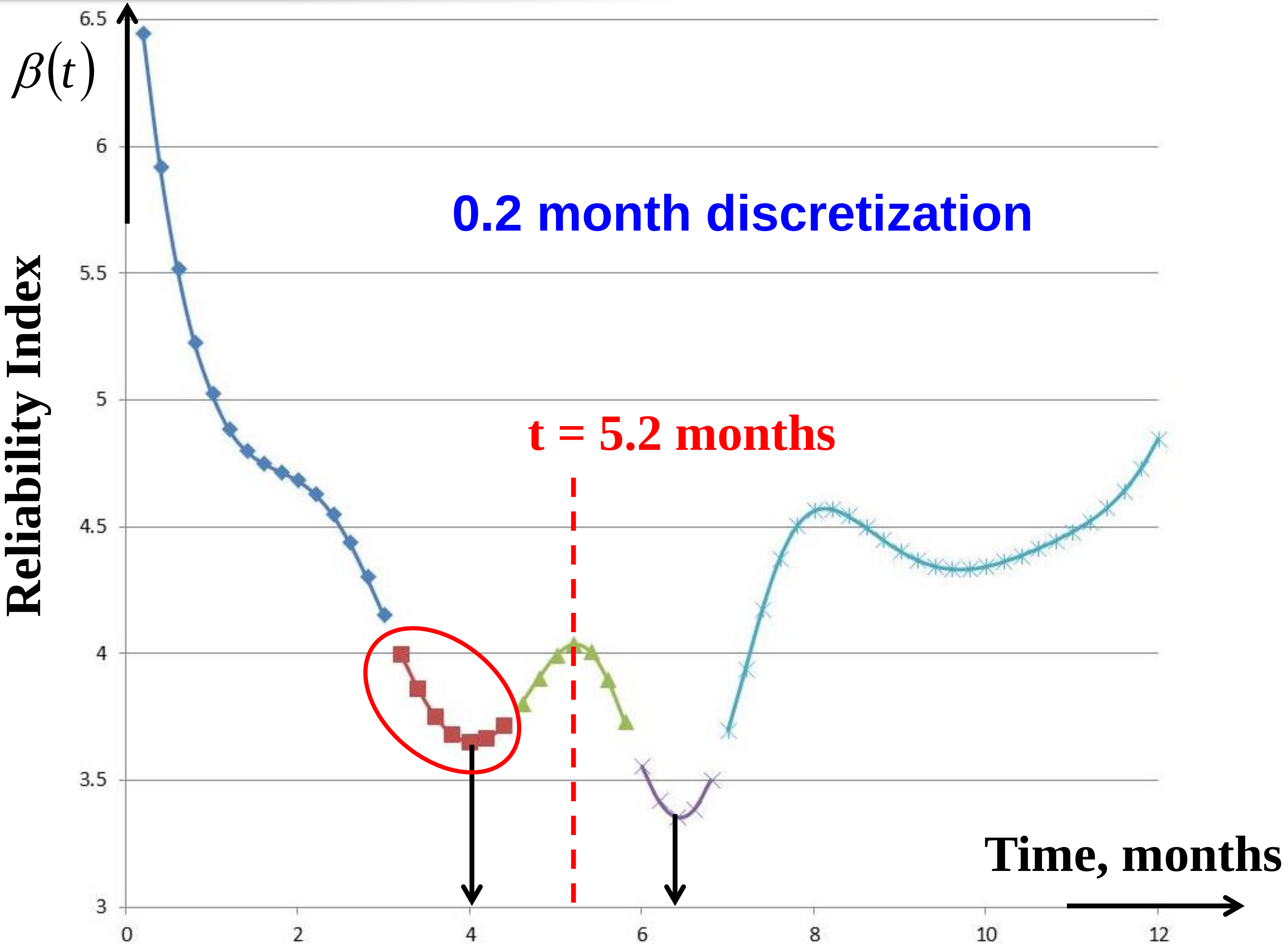


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Illustration of Composite Limit State

0.1 month discretization

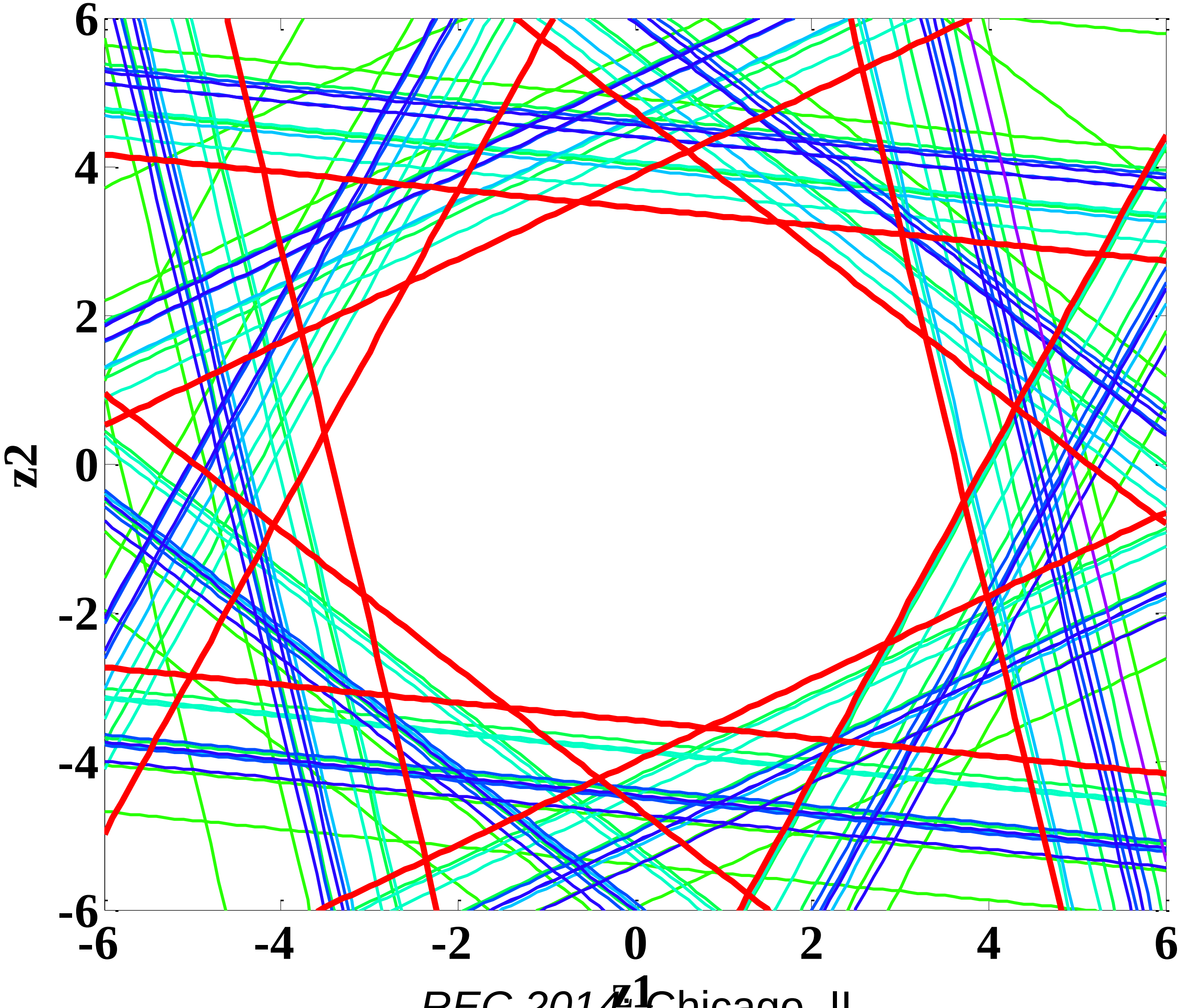


Illustration of Composite Limit State

0.05 month discretization

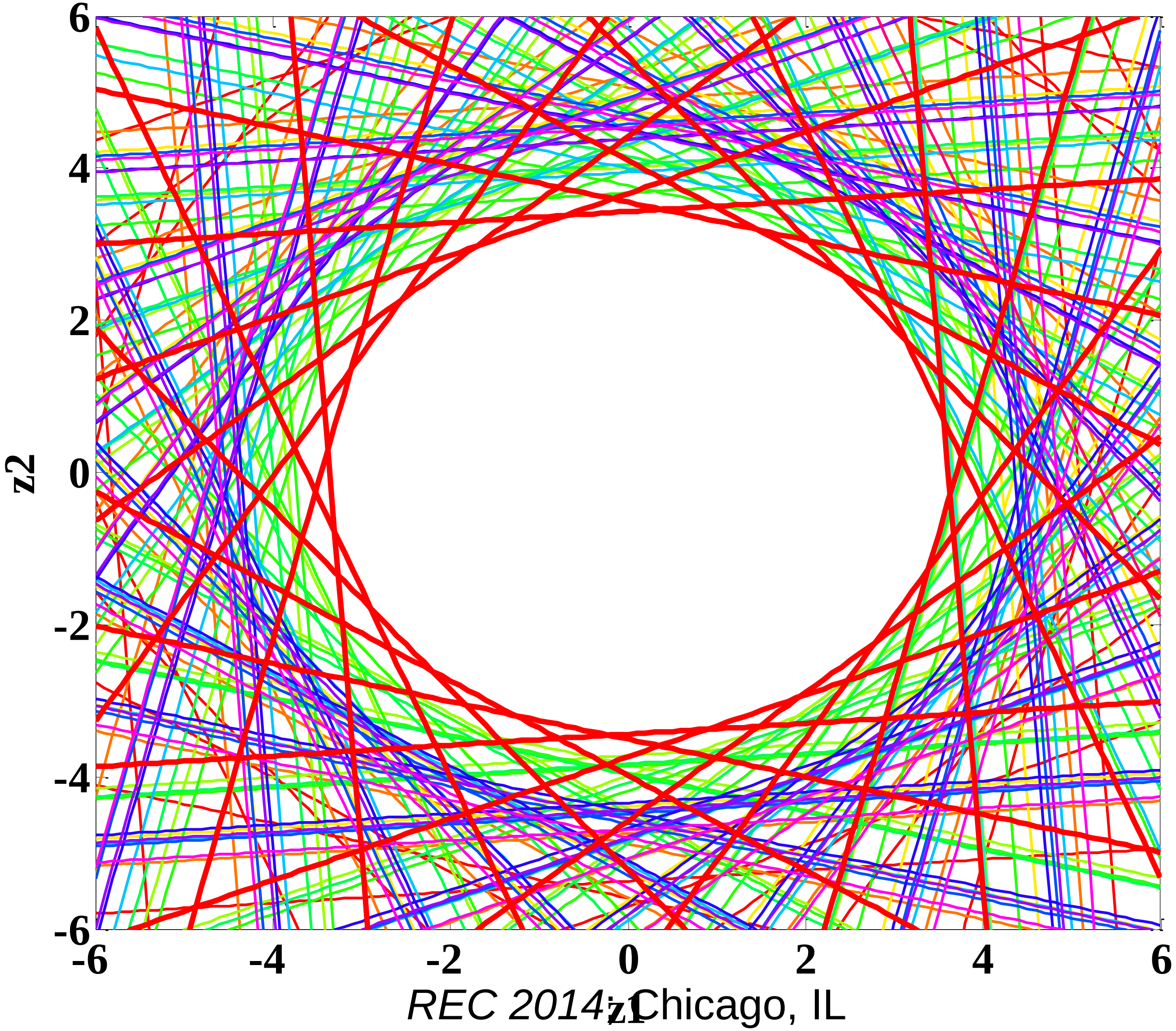
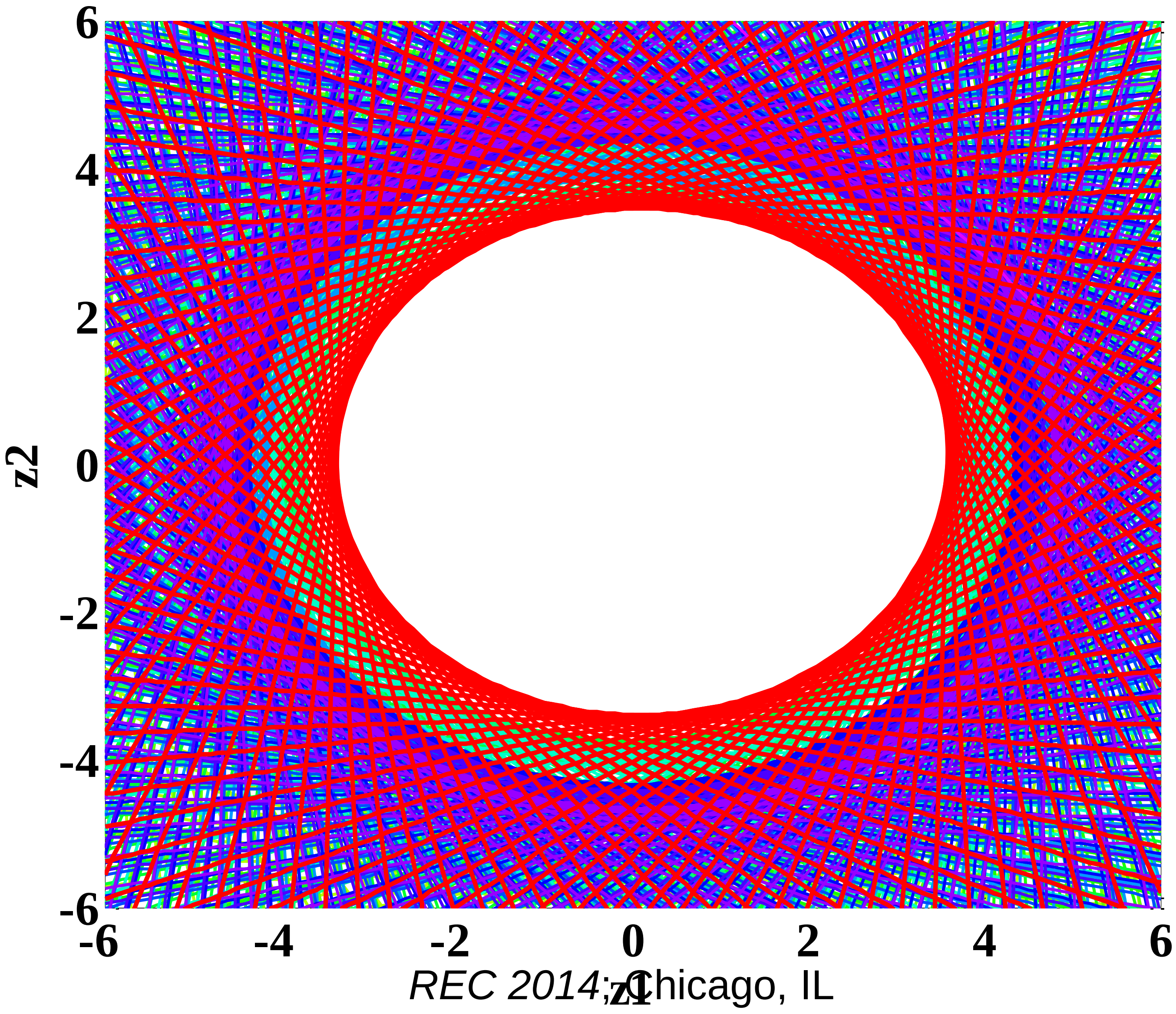
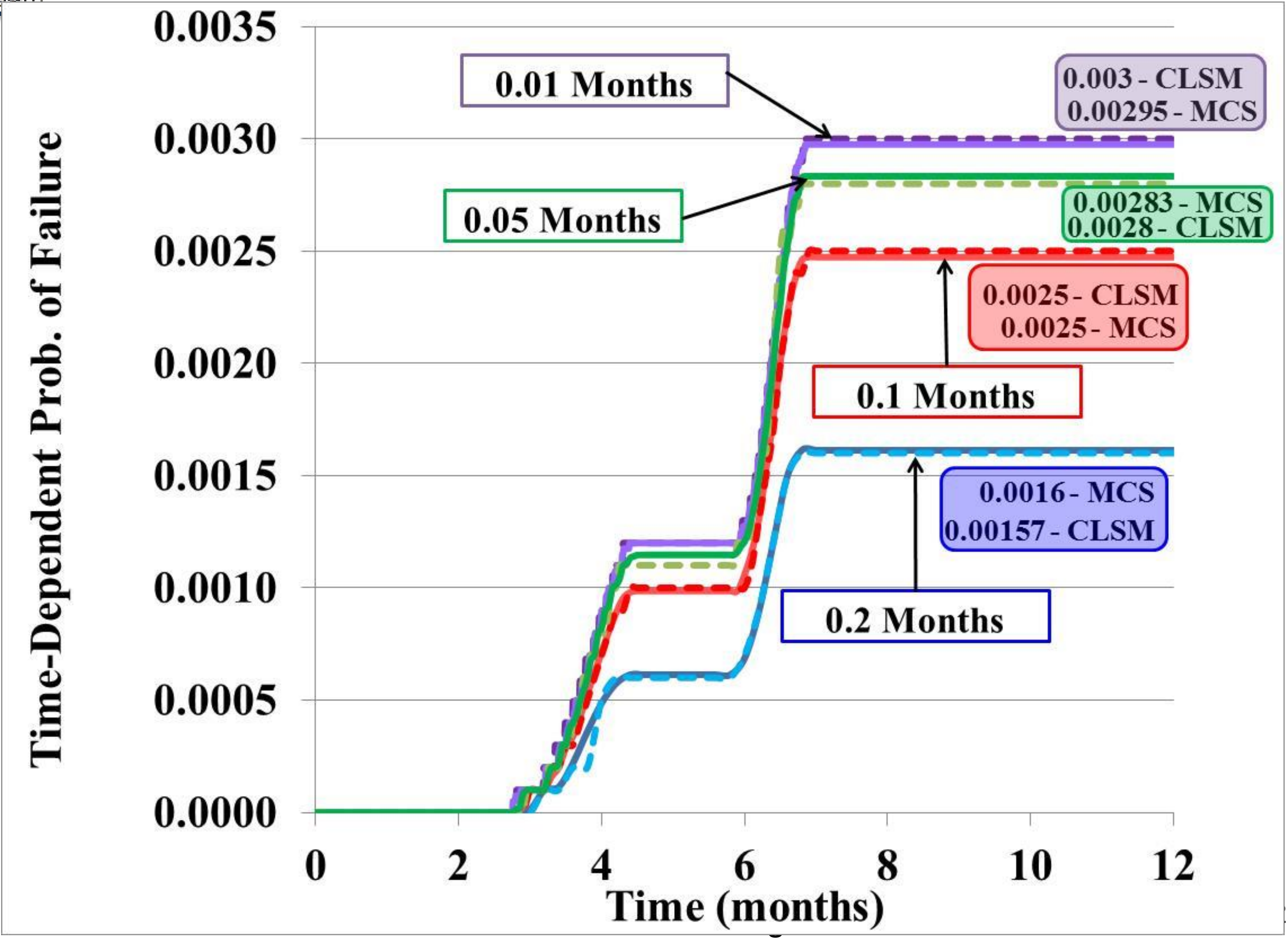


Illustration of Composite Limit State

0.01 month discretization



Probability of Failure Calculation



Implementation Points

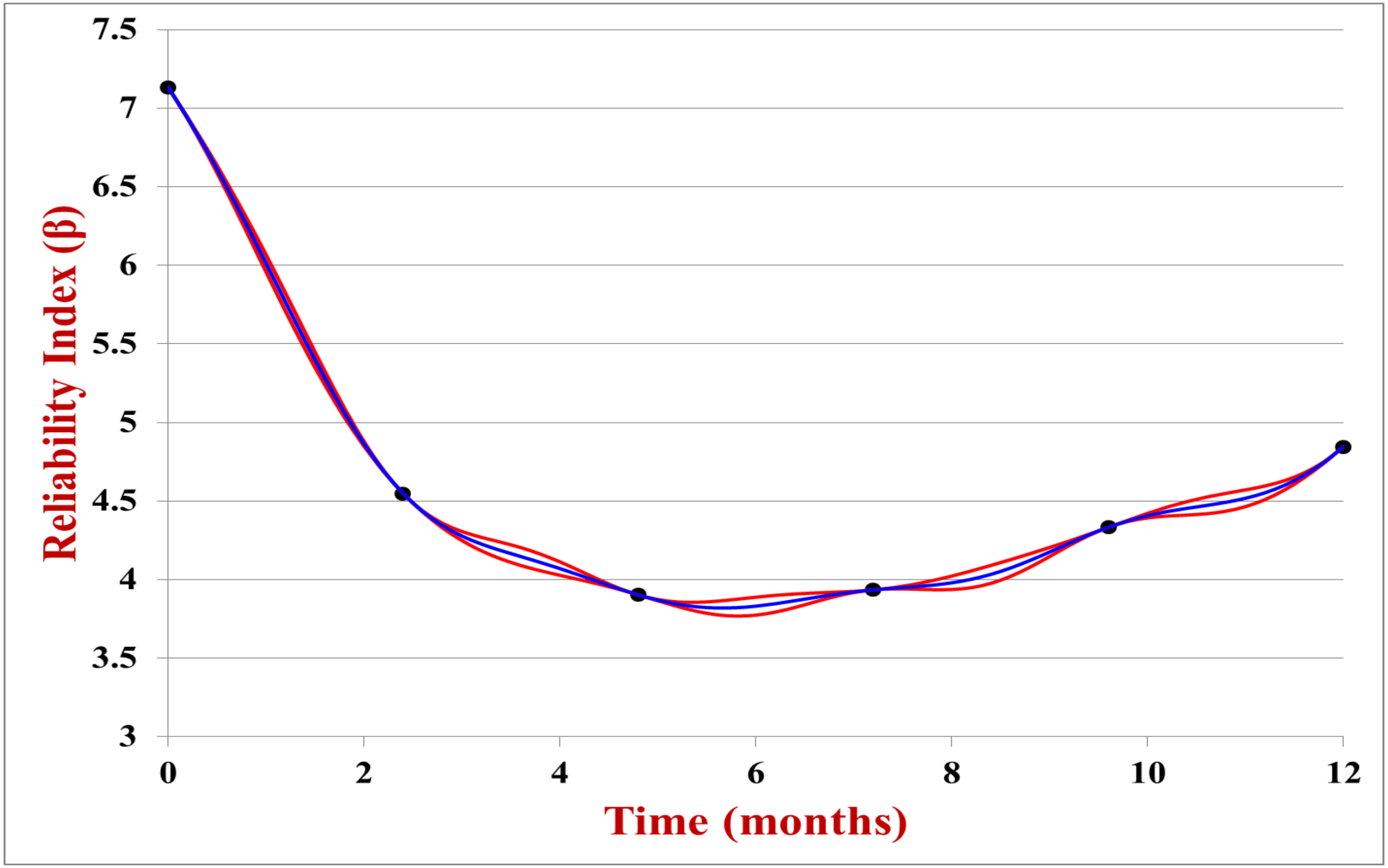
Observations:

- Proposed approach requires a time-independent analysis (beta and MPP) at ALL time steps.
- Only low-beta limit states contribute to composite limit state.

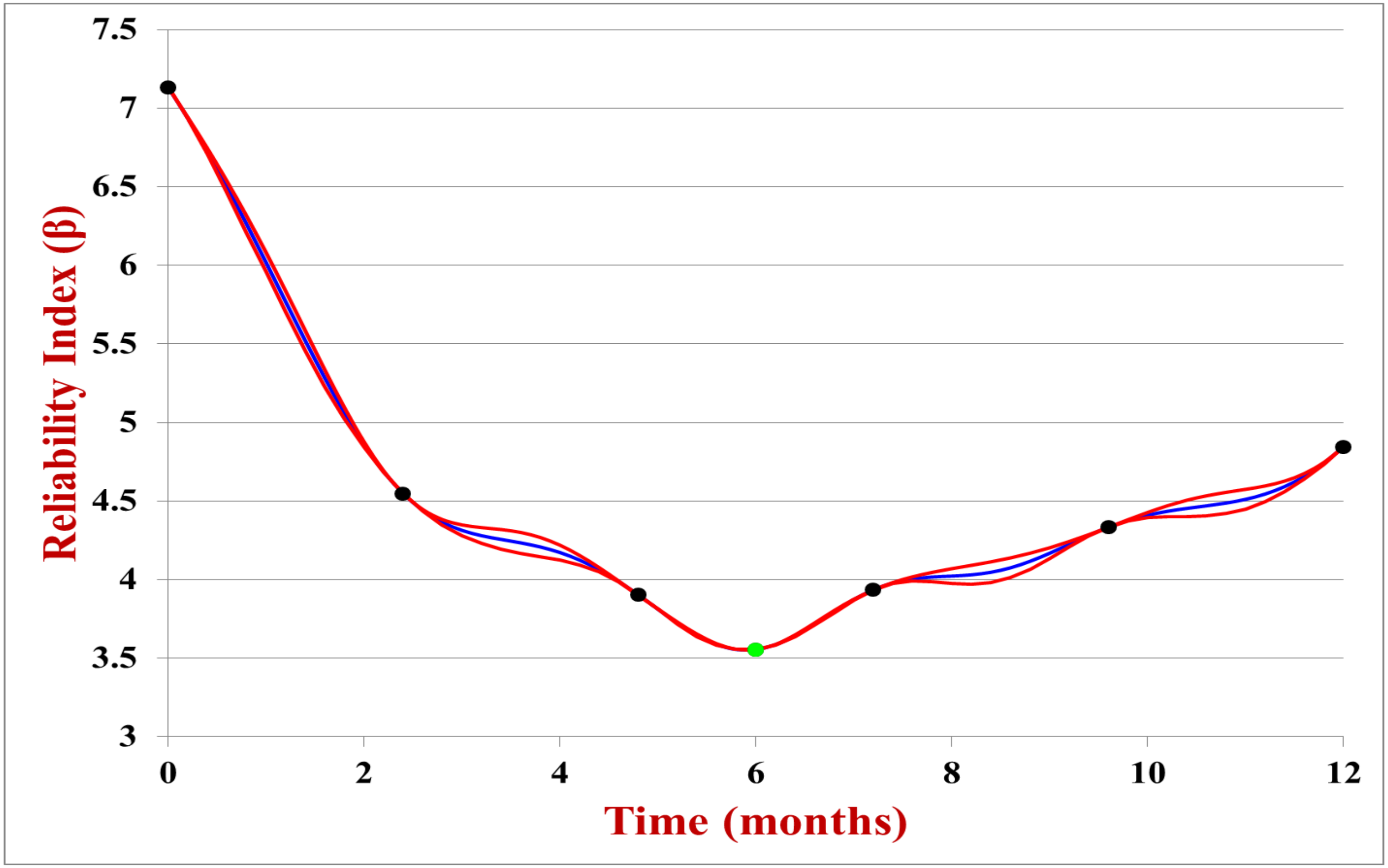
➢ To avoid calculating beta and MPP at ALL time steps:

  - Build a surrogate of beta curve using Kriging
  - Build composite limit state (CLS) progressively starting with times where beta is low.
  - Stop if CLS does not change further.

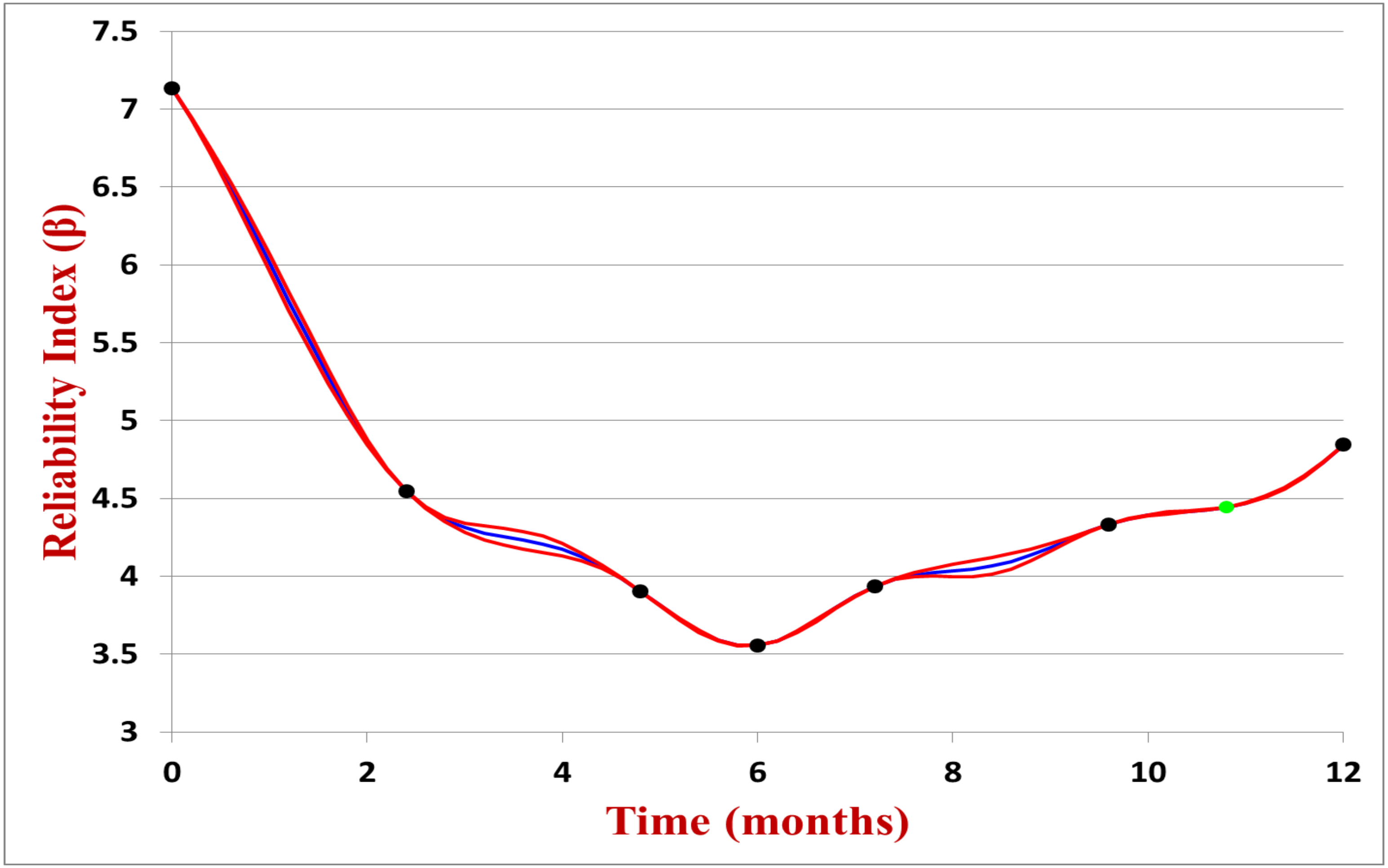
Reliability Index Estimation using Kriging



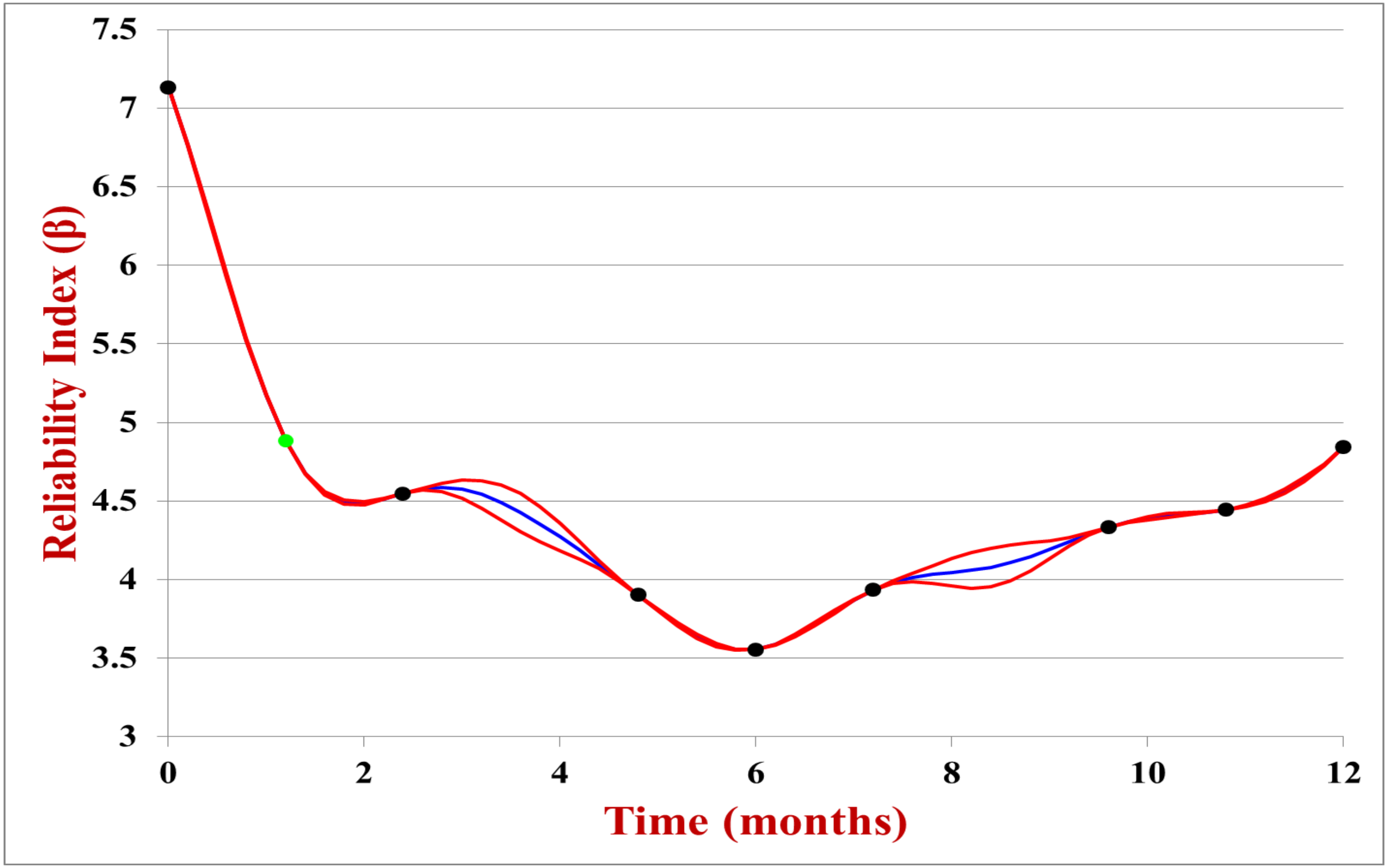
Reliability Index Estimation using Kriging



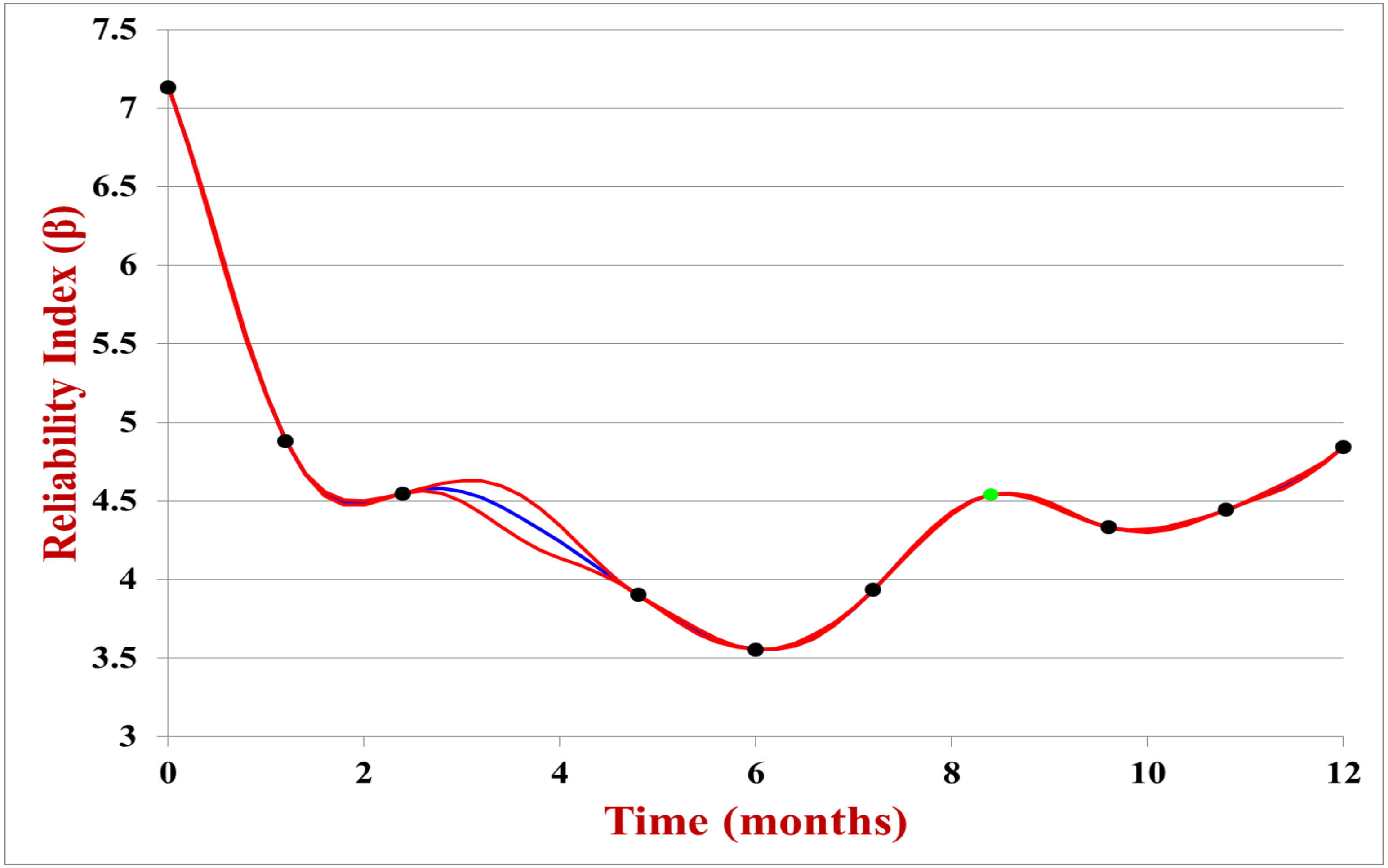
Reliability Index Estimation using Kriging



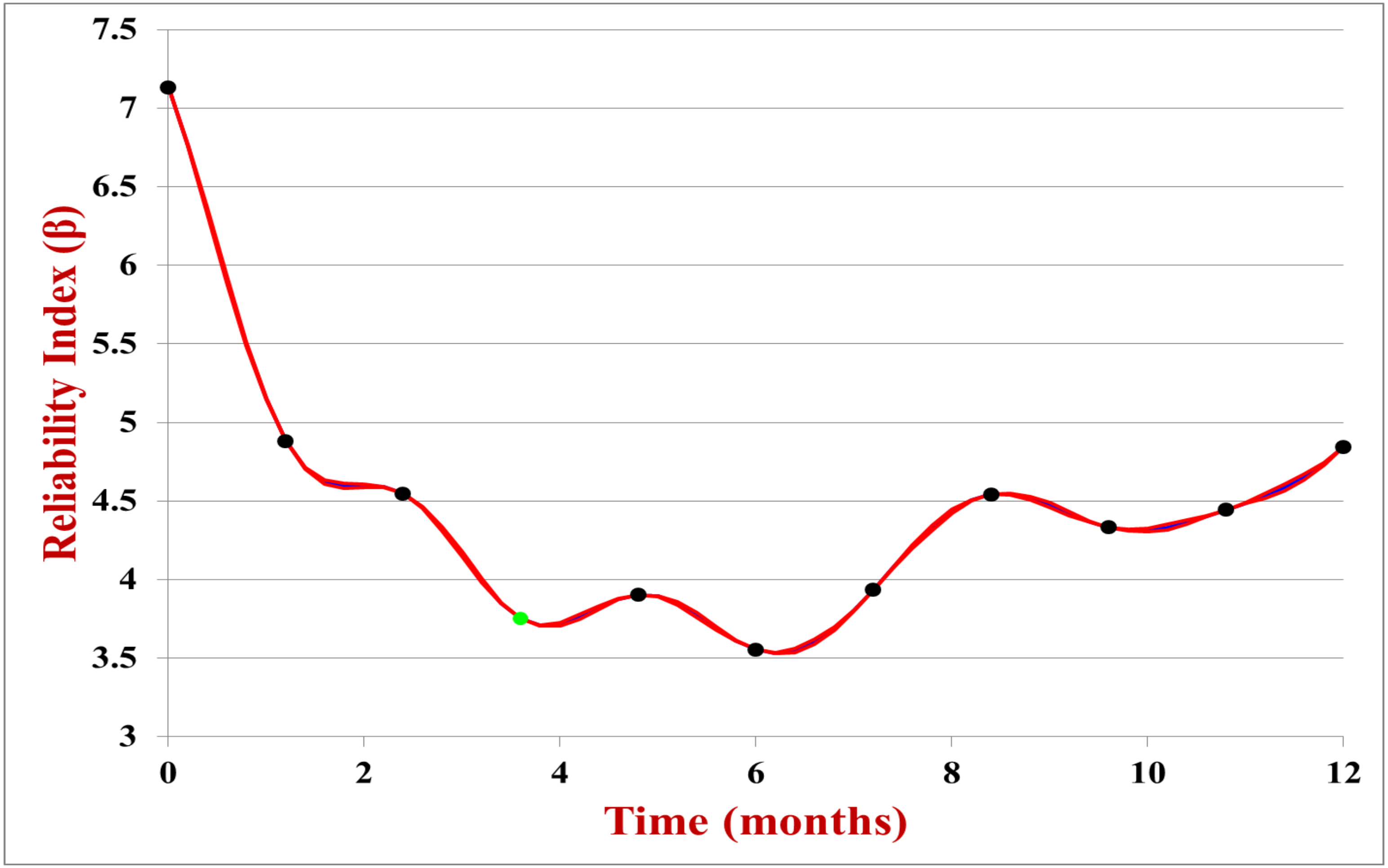
Reliability Index Estimation using Kriging



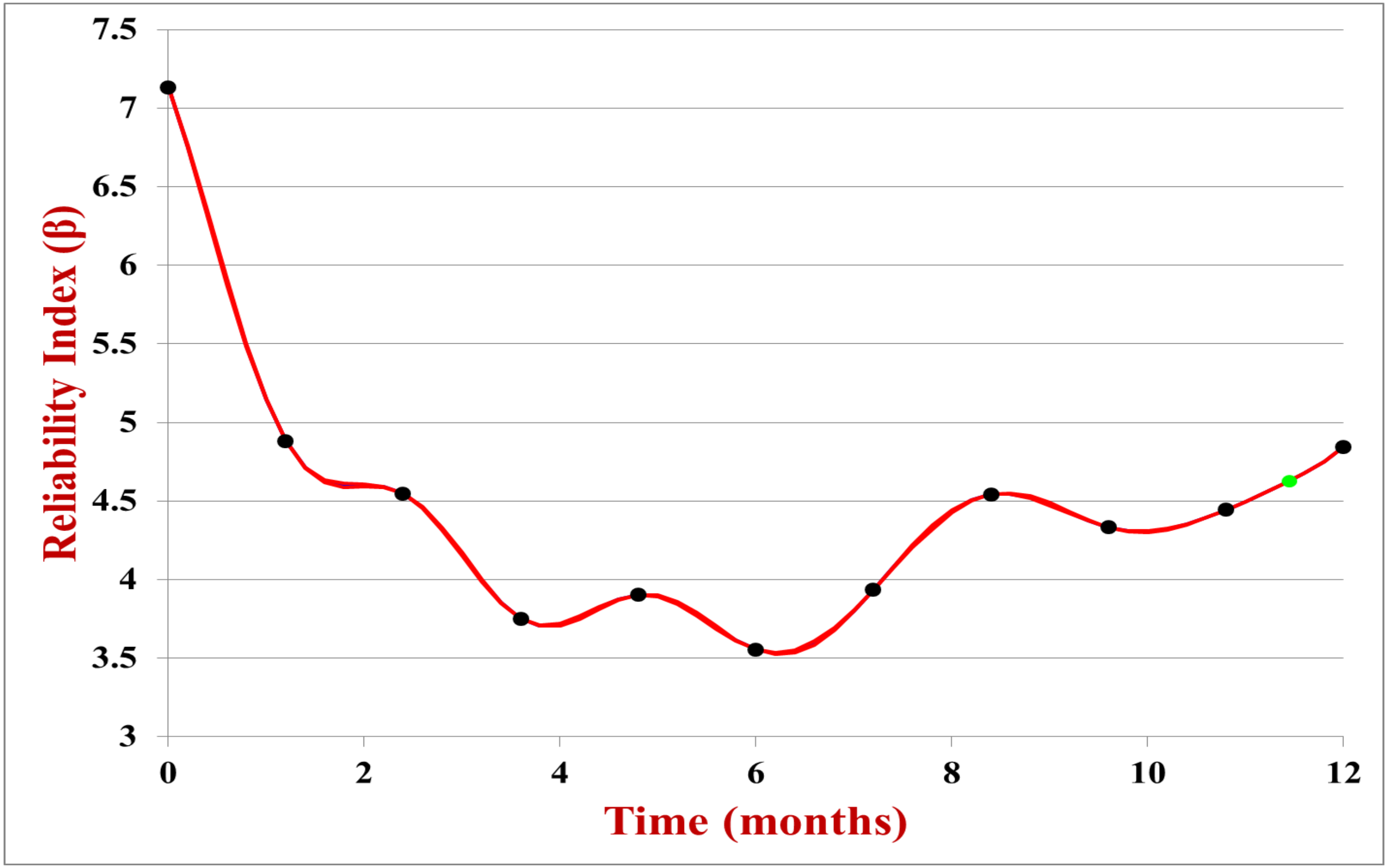
Reliability Index Estimation using Kriging



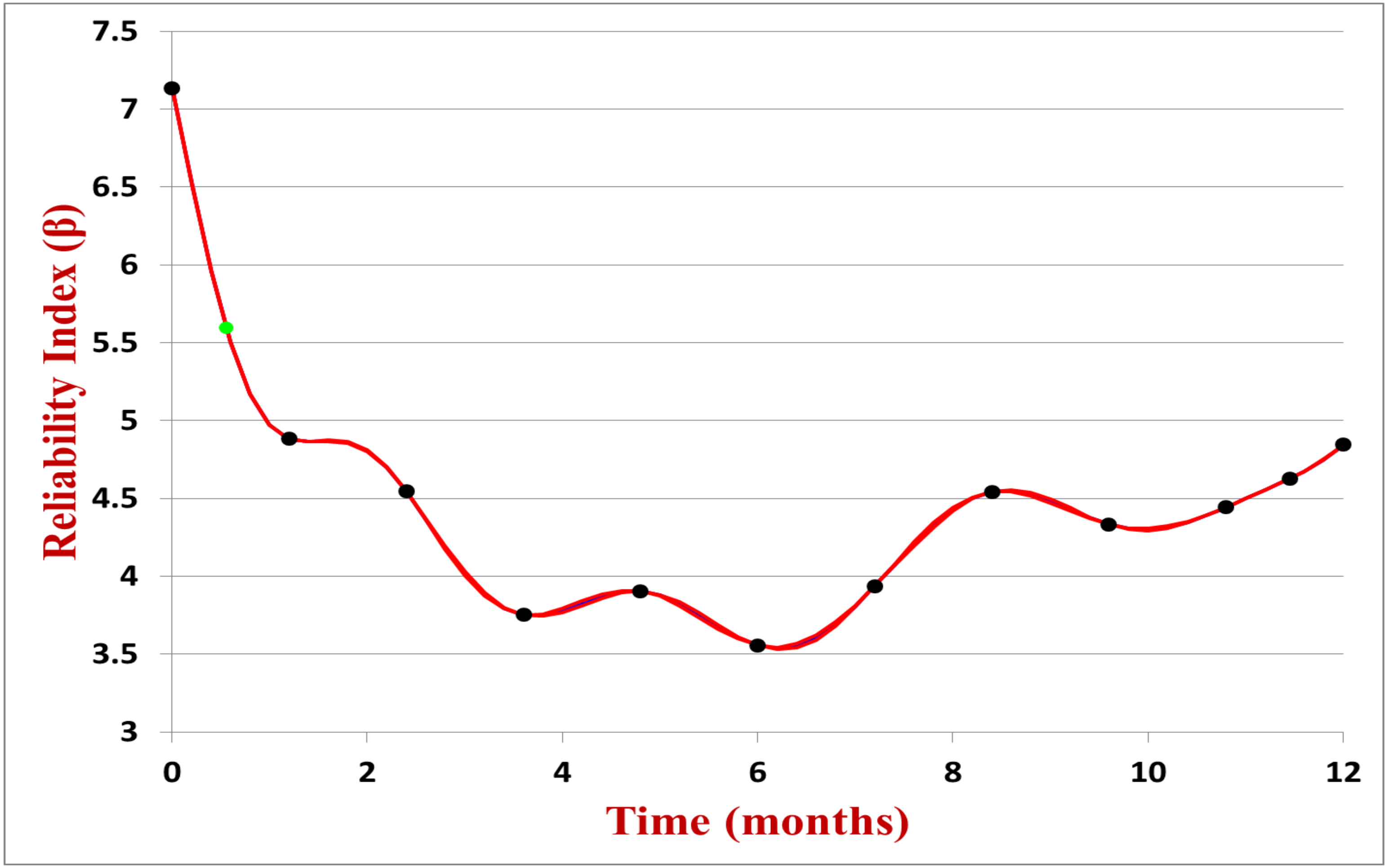
Reliability Index Estimation using Kriging



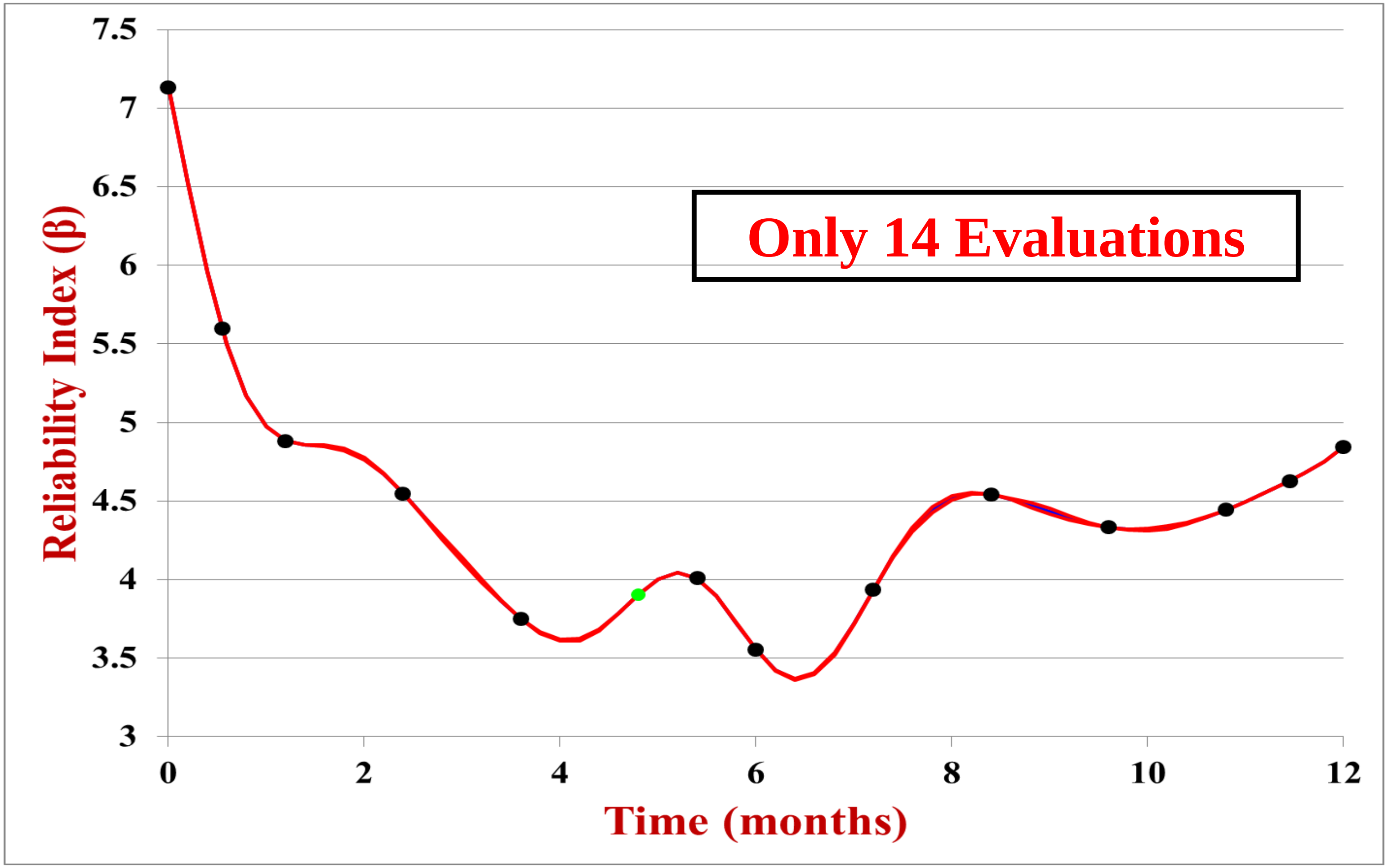
Reliability Index Estimation using Kriging



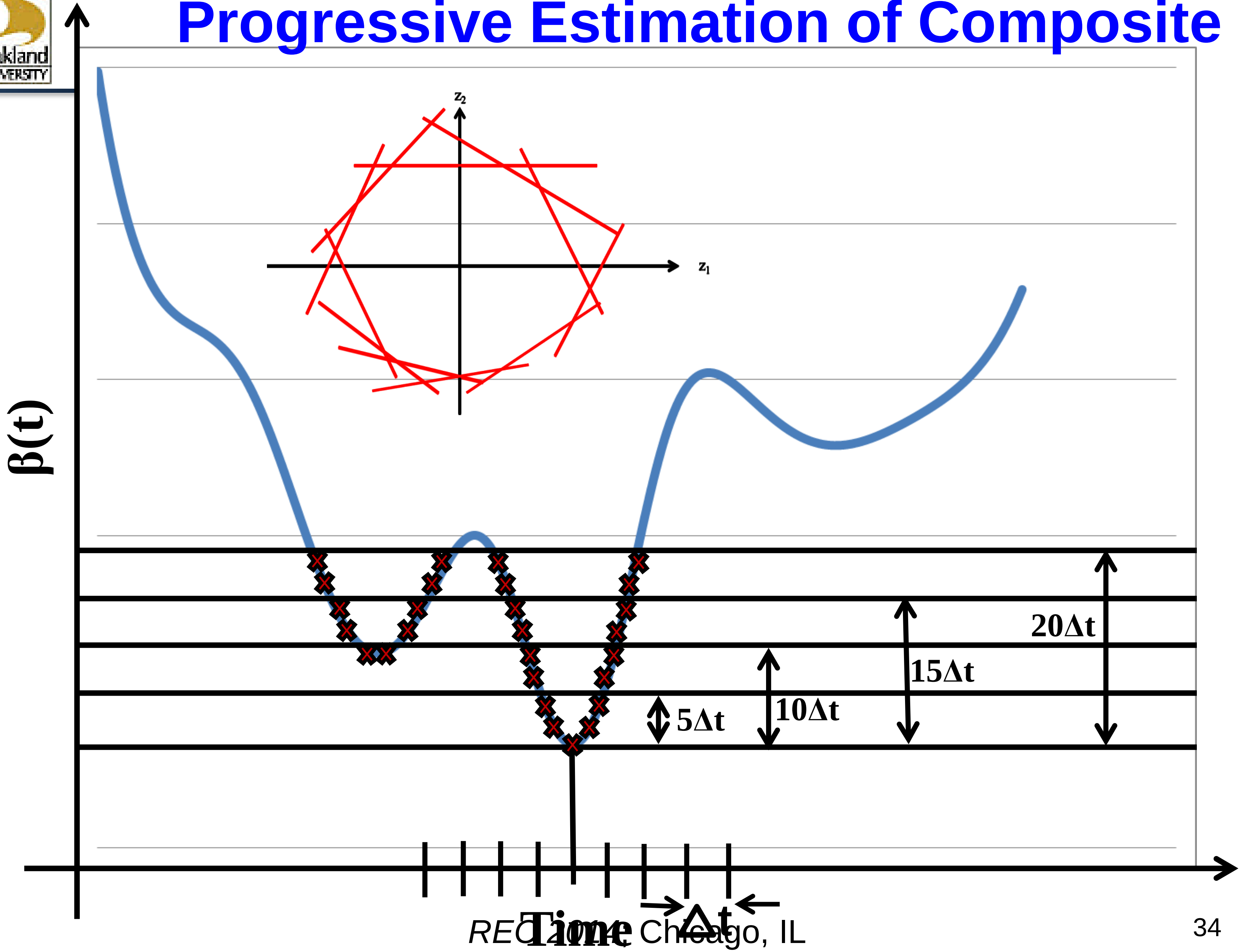
Reliability Index Estimation using Kriging



Reliability Index Estimation using Kriging



Progressive Estimation of Composite



Total Probability Theorem

- Proposed approach is based on FORM.
- FORM's accuracy deteriorates if
  - Limit state is nonlinear
  - Random variables are non-normal and / or correlated.
- The Total Probability Theorem can increase accuracy

$$P_f(0,T) = P(F) = \int_{\Omega} P\left(F/\mathbf{W}\right) f_{\mathbf{W}}(\mathbf{w})\, d\mathbf{w}$$

Mean value of conditional probability

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Total Probability Theorem

Example :

$$g(\mathbf{X},\mathbf{Y}(t),t)=X_1^3+2X_1X_2^2+\left(X_3t^2+X_4^2t\right)+X_1Y_2(t)^2+3X_5Y_1(t)\le 0$$

$Y_1(t), Y_2(t)$: Gaussian Processes

X_2, X_4, X_5 : Non-normal R.V.s

If $\mathbf{W}=\left[X_1 \quad X_2 \quad X_4 \quad X_5 \quad Y_2(t)\right]$, then $P\left(F/\mathbf{W}\right)$ is calculated using

$$g(\cdot,t)=x_1^3+2x_1x_2^2+\left(X_3t^2+x_4^2t\right)+x_1y_2(t)^2+3x_5Y_1(t)$$

Realization of X_1

Normal R.V.

Function of Normal R.V.s

Linear function of Normal R.V.s

Key Points of Composite Limit State Method

- **Efficient estimation of reliability index $\beta(t)$**
  - ➢ **Kriging metamodel using prediction variance estimation**
- **Identification of Composite Limit State using $\beta(t)$**
  - ➢ **Convex domain formed by instantaneous limit states with “smallest” betas**
- **Calculation of P_f using Composite Limit State**
  - ➢ **Exact calculation using the convex polyhedron of the safe domain**

Summary and Future Work

- Developed a new time-dependent reliability estimation method using the concept of Composite Limit State (CLS). It
  - Identifies the CLS automatically
  - Calculates time-dependent probability of failure exactly for convex linear safe sets

Future Work

- Improve computational efficiency (e.g. use Kriging to estimate MPP locus)
- Apply method to estimating remaining life due to fatigue failure

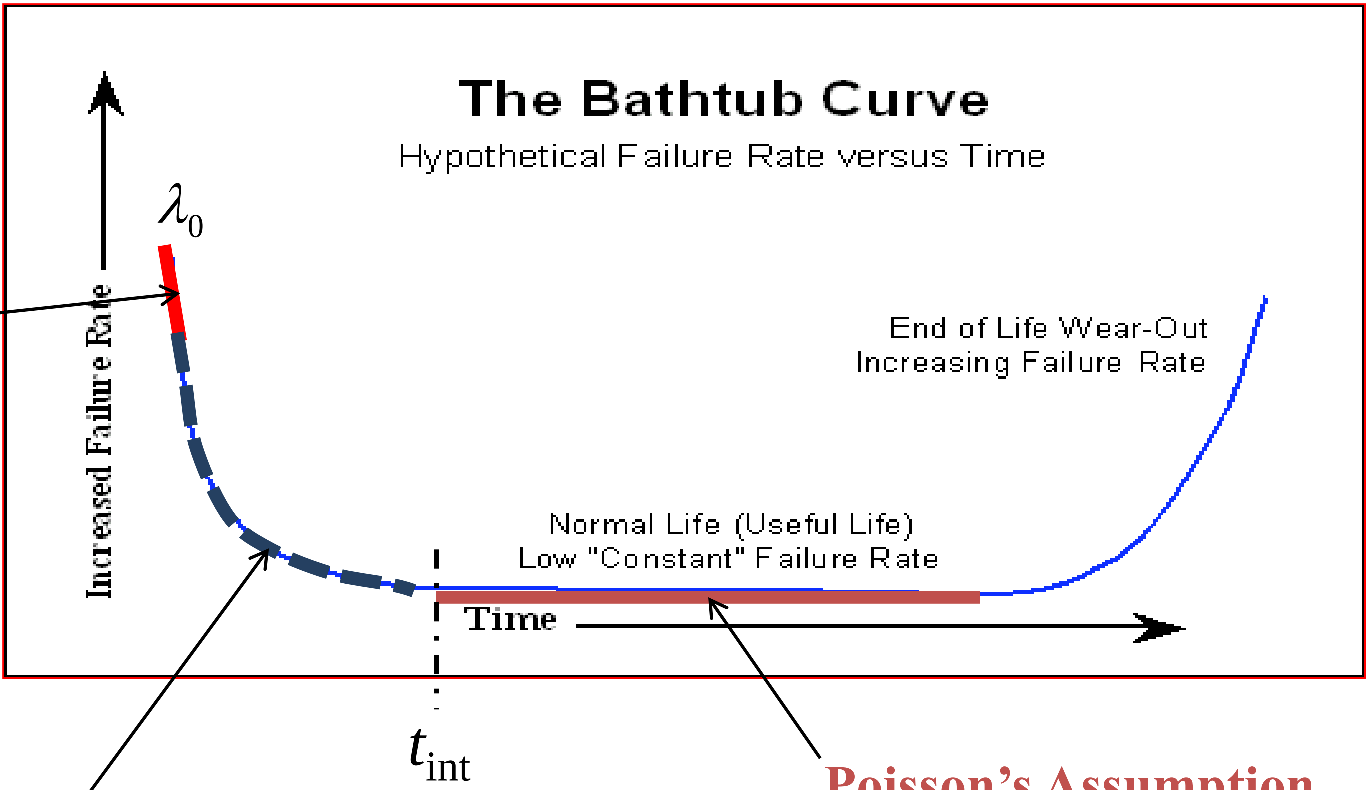
Thank you for your attention

Our Approach

A novel MC-based method to calculate the time-dependent reliability (cumulative probability of failure) based on :

- **short-duration data and an exponential extrapolation using MCS or Importance Sampling (Infant Mortality)**
- **Poisson's assumption (Useful Life)**

Efficient MC Simulation Approach



MCS

$$b = -\frac{1}{\lambda_0}\left(\frac{d\hat{\lambda}}{dt}\right)_{t=0}$$

Exponential Extrapolation

$$\boxed{\hat{\lambda}(t) \approx \lambda_0 e^{-bt}}$$

Poisson's Assumption

$$\boxed{F_T^c(t) = \begin{cases} 1-e^{-\int_0^t \hat{\lambda}(t)dt} & ,t \in [0, t_{\text{int}}] \\ 1-(1-F_T^c(t_{\text{int}}))e^{-\nu_m(t-t_{\text{int}})} & ,t \in [t_{\text{int}}, t_f] \end{cases}}$$

Objectives and Scientific Contributions

Objectives for Project 5.3

- Develop a methodology to estimate reliability and remaining life of a vehicle system using time-dependent reliability / durability principles.
- Use the methodology to improve existing accelerated Life Testing (ALT) methods by
  - Shortening testing time, and
  - Using realistic testing conditions
- Implement all developments in TARDEC's Physical Simulation Lab

Fundamental Scientific Contributions

- Developed advanced statistical methods to calibrate a math model using a limited number of tests.
- Developed a novel time-dependent reliability method using the concept of composite limit state. The method also advances state-of-the-art in system reliability
- Developed a new paradigm for Accelerated Life Testing

Accelerated Life Testing (ALT)

Relates reliability measured under high stress conditions to expected reliability under normal conditions

Our Goal: Shorten testing time and use realistic testing conditions

Advantage:

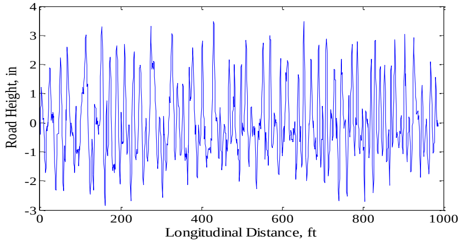
- ➢ Shortens testing time

Disadvantage:

- ➢ Uses unrealistic testing conditions

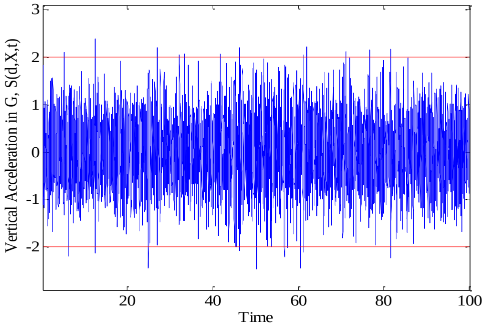
Problem Description

Random Variables



Terrain, Engine Load, etc.

Durability/Performance Measures in Time

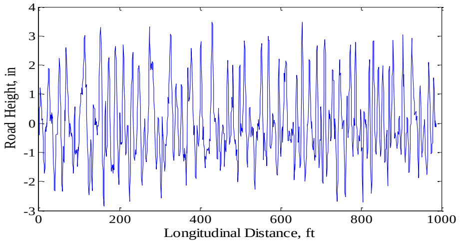


Vehicle speed : 20 mph; Mission distance : 100 miles

Simulation can be practically performed for a short-duration time

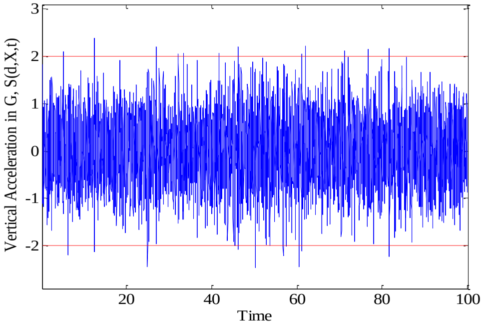
Observations / Challenges

Random Variables



Terrain, Engine Load, etc.

Durability/Performance Measures in Time



<u>Major Challenges:</u>

- **Model input random processes (terrain, engine load)**
- **Develop detailed, and yet simple and accurate vehicle math models**
- **Run math models for long time**

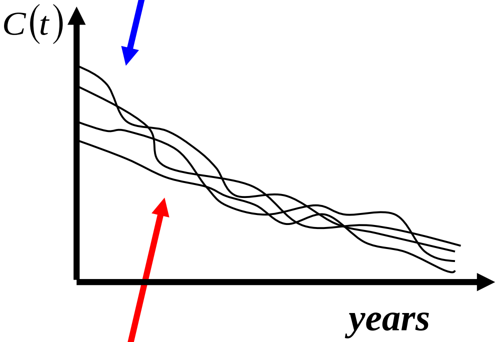
Our Approach to Address Challenges

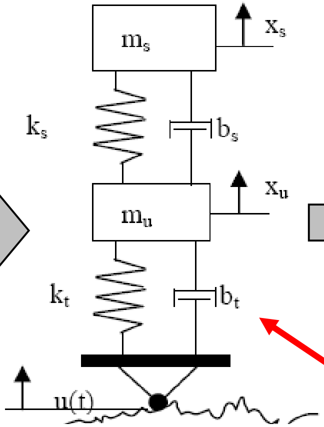
- **Model input random processes (terrain, engine load)**
  - **Time series and spectral decomposition methods**
- **Develop detailed, and yet simple and accurate vehicle math models**
  - **Use available math models**
  - **Calibrate math models using tests to improve their accuracy**

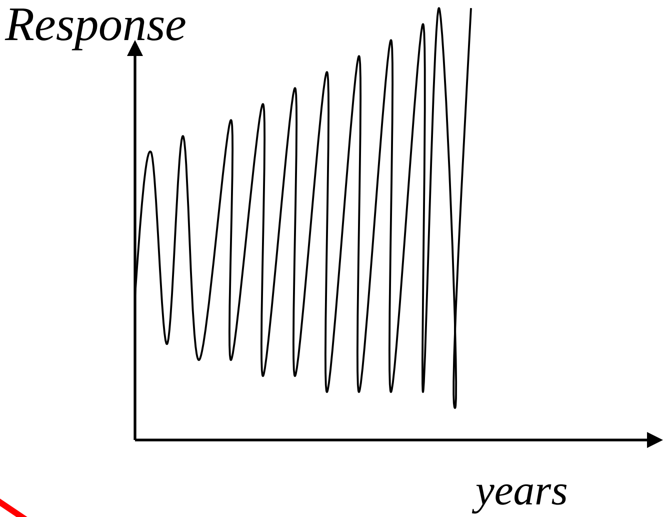
    (Model V&V approach)
- **Run math models for long time**
  - **Run calibrated math models for a short duration**

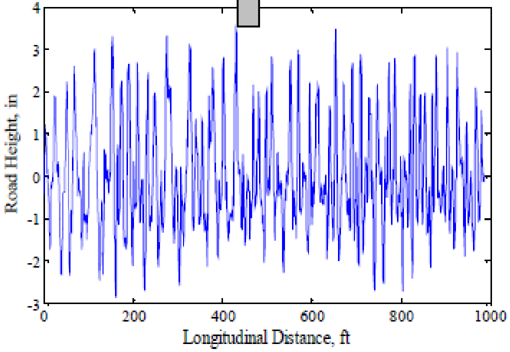
Proposed Approach for ALT

Degraded vehicle parameter (e.g. k_s or b_s)









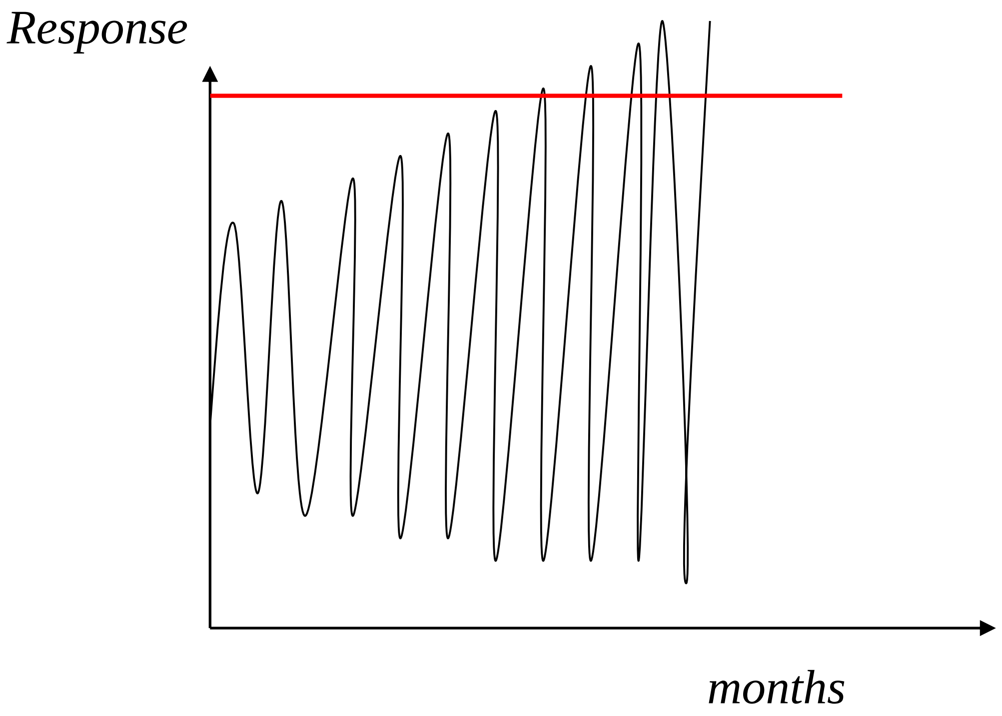
Random process characterized by time series

Available math model calibrated using tests

Main Task in Proposed ALT

Calculate reliability through time

Reliability at time *t* is the probability that the system has not failed before time t.



Must calculate time-dependent probability of failure

Definitions / Observations

Reliability: **Ability of a system to carry out a function in a time period [0, t_L]**

$$p_f^c = P(t \le t_L) = F_T^c(t_L)$$

Prob. of Time to Failure

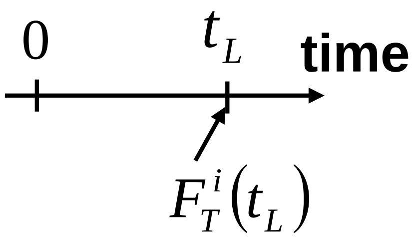
$$\boxed{F_T^c(t_L) = P\left(\exists t \in [0, t_L], such \quad that \quad g(\mathbf{X}(t), t) \le 0\right)}$$

Cumulative Prob. of Failure

$$\boxed{F_T^i(t_L) = P\left(g(\mathbf{X}(t_L), t_L) \le 0\right)}$$

Instantaneous Prob. of Failure

Time-Invariant Reliability



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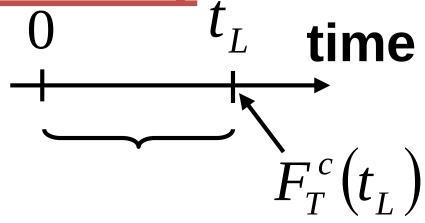
Time-Variant Reliability



Definitions / Observations

Reliability: **Ability of a system to carry out a function in a time period [0, t_L]**

$$p_f^c = P(t \le t_L) = F_T^c(t_L)$$ **Prob. of Time to Failure**

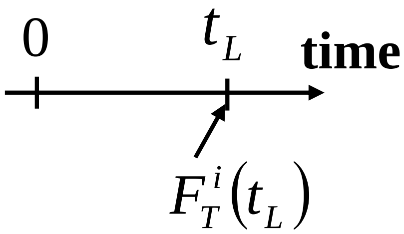
$$\boxed{F_T^c(t_L) = P\left(\exists t \in [0, t_L], such \quad that \quad g(\mathbf{X}(t), t) \le 0\right)}$$

Cumulative Prob. of Failure

$$\boxed{F_T^i(t_L) = P\left(g(\mathbf{X}(t_L), t_L) \le 0\right)}$$

Instantaneous Prob. of Failure

Time-Invariant Reliability **Time-Variant Reliability**



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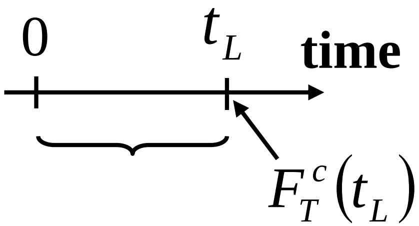
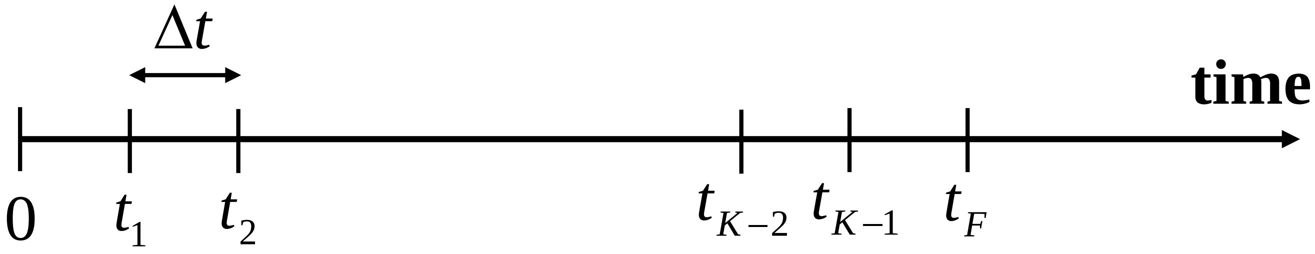


Calculation of Cumulative Probability of Failure

- **State-of-the-art Approaches**
  - **PHI2 method (Andrieu-Renaud, et al., *RESS*, 2004)**
  - **Set-Based approach (Son and Savage, *Quality & Rel. Engin.*, 2007)**



$$t_F = K\Delta t$$

- **State-of-the-art approaches are in general, inaccurate due to:**
  - **Choice of** Δt
  - **Not including contribution of <u>all</u> discrete times**

What is Reliability?
Cumulative Probability of Failure

Reliability at time t is the probability that the system has not failed before time t.

$$F_T^c(t_L) = P\left(\exists t \in [0, t_L], such \quad that \quad g(\mathbf{X}(t), t) \le 0\right)$$

Cumulative Prob. of Failure

$$F_T^i(t_L) = P\left(g(\mathbf{X}(t_L), t_L) \le 0\right)$$

Instantaneous Prob. of Failure

Calculation Methods for $F_T^c(t)$

- Maximum Response Method
- Niching GA & Lazy Learning Local Metamodeling
- MCS / Importance sampling

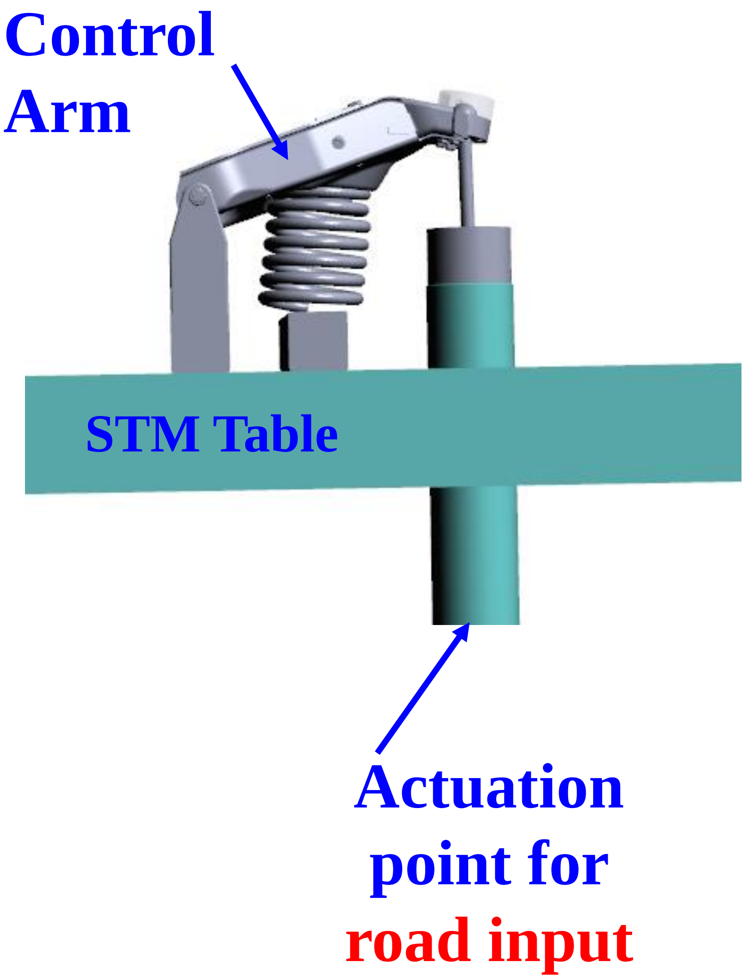
} Analytical

$$F_T^c(t) = 1 - \exp[-\int_0^t \lambda(t)\, dt]$$

} Simulation-based

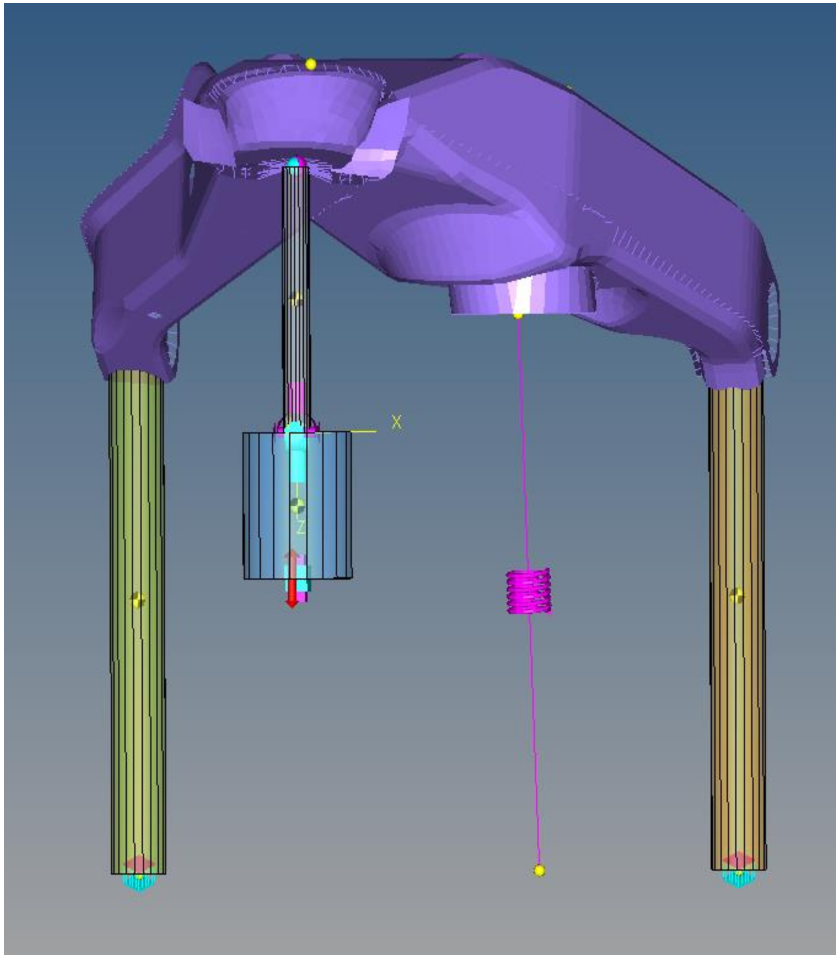
Test Fixture at TARDEC

HMMWV Control arm / Spring



Output : Stress or Strain at different locations on control arm

HMMWV Lower Control Arm Fixture



MotionView Math Model

Physical Model

Key Points of Proposed Approach

Time series and Spectral Decomposition

- Characterization of random processes

Composite Limit State Approach

- Calculation of time-dependent probability of failure (or reliability)

Companion project

- Model validation through calibration

$$\boxed{y^{t}\left(\mathbf{x},\tau\right)=y^{m}\left(\mathbf{x},\mathbf{c},\tau\right)+\delta\left(\mathbf{x},\tau\right)+\varepsilon\left(\mathbf{x},\tau\right)}$$

Prediction bias

Zero mean random error