Geometric Misfitting in Structures – An Interval-Based Approach

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Abstract: In this work, the issue of predicting structural behaviour in the present of geometric uncertainty arising out of fabrication errors and/or thermal changes in engineering systems is addressed. Geometric uncertainty is expressed as the deviations of actual dimensions of the system components from their corresponding nominal dimensions (misfitting) and is expressed in interval form. Such geometric uncertainty is converted into an equivalent nodal load uncertainty. The present work eliminates the element force overestimation resulting in our previous formulation (Muhanna, et al, 2006). The mixed interval finite element formulation developed by the authors (Rama Rao, Mullen and Muhanna, 2010) is utilized to obtain sharp bounds to displacements and element forces in the presence of geometric, material and load uncertainties. The present work also computes an exact enclosure to the final system geometry. Results are illustrated in example problems.

Keywords: Interval; Interval Finite Elements; Uncertainty; Misfitting; Geometry.

1. Introduction

Engineering systems are usually designed with a pre-described geometry in order to meet the intended function for which they are designed. However, due to fabrication errors and/or thermal changes, the dimensions of system components will deviate from their nominal values creating a misfitting problem during the manufacturing/construction process. In engineering practice, such a fabrication deviation is defined in a form of maximum allowable tolerance for individual components or for the completed system after the assemblage. Usually, the design and manufacturing processes of mechanical components require a complete definition of geometry of these components, however the definition of the geometries of the components are only considered complete if tolerances are included in the design. Thus, a proper design methodology should be capable of incorporating the tolerances to geometric properties in design. This means that geometries of the geometrical elements of a work-piece are completely defined and tolerances on fabrication errors are properly accounted for (Henzold, 1995).

State of the art technologies are striving for higher performance, higher efficiency and greater reliability. To achieve such goals, the analysis and design procedures have to account for all possible factors that could affect the product. Tolerances represent one of the main sources of uncertainty that should be accounted for.

Tolerances, usually, are defined as absolute deviation from the nominal values. Thus, including the tolerance, in the analysis and design, as a possible value within a given interval that possesses known bounds might be a realistic or natural way of representing such type of uncertainty. More information about the allowable values of fabrication tolerances can be found in the publications of international organizations such as the International Standards Organization (ISO/IEC,2008).and in national organizations such as the Bureau of Indian Standards (IS 7215 ,1974) In the present work, tolerances (geometrical uncertainty) will be introduced as interval values i.e., the true value is known to lie between two bounding values, but the exact value is unknown. This calls for the use of Interval Finite element methods.

An initial effort to obtain sharp bounds to structural response of plane trusses to geometric uncertainty by interval finite element approach was made earlier by the authors and dependency problem has been addressed by the M matrix approach (Mullen and Muhanna,1999; Muhanna, Erdolen and Mullen, 2006). The problem of obtaining axial forces of the truss at the same level of sharpness as displacements is handled in the present work by following an earlier mixed interval finite element formulation developed by the authors (Rama Rao, Muhanna and Mullen, 2011). This formulation is described in section 3 and is applied in the case of simultaneous presence of load, geometric and material uncertainties.

2. A Review of Interval finite element method

Since the early development of Interval Finite Element Methods (IFEM) during the mid nineties of last century, several researchers worked on various aspects of Interval Finite Element Methods. (Alefeld and Herzberger, 1983; Neumaier, 1987; Köylüoglu, Cakmak, Ahmet, Soren, 1995; Koyluoglu, Cakmak and Nielsen,1995; Rao and Sawyer, 1995; Muhanna and Mullen,1995; Nakagiri and Yoshikawa,1996; Mullen and Muhanna,1996, Rao and Berke,1997;Koyluoglu, and Elishakoff, 1998; Rao and Chen, 1998,Nakagiri and Suzuki,1999; Muhanna and Mullen,1999; Mullen and Muhanna,1999). In particular, researchers have focused among other issues on two major problems; the first is how to obtain solutions for the resulting linear interval system of equations with reasonable bounds on the system response that make sense from practical point of view, or in other words with the least possible overestimation of their bounding intervals, the second is how to obtain reasonable bounds on the derived quantities that are functions of the system response. For example, when the system response is the displacement, the derived quantities might be forces or stresses which are functions of the displacements. Obtaining tight bounds on the derived quantities has been a tougher challenge due to the existing dependency of these quantities on the primary dependent variables which are already overestimated.

During the last decade, several significant advances have been made in the application of Interval Finite Element Methods (IFEM) to problems of uncertainty structural mechanics. Important contributions have been made by several researchers during the past decade (Muhanna and Mullen, 2001;Corliss, Foley and Kearfott, 2004; Zhang, 2005; Muhanna, Zhang and Mullen, 2005; Popova, Iankov, Bonev, 2006;Neumaier and Pownuk, 2007;, Rama Rao, Mullen and Muhanna, 2011). These researchers focused on the issue of computing structural response in the presence of uncertainty load, stiffness, and element cross sectional area. However, the uncertainty in the components’ length has not been addressed till the initial study in this direction was made by the authors (Muhanna, Erdolen and Mullen, 2006).

A significant effort has been made in the work of Zhang (2005) to control the additional overestimation in the values of the derived quantities; the derived quantities have been calculated by an implicit substitution of the primary quantities. In spite of the advancement provided by this approach, still it is conditioned by the original IFEM formulation and the special treatment of required transformations.

A significant improvement in the formulation of IFEM with application to truss problems has been introduced in the work of Neumaier and Pownuk (2007). This work has presented an iterative method for computing rigorous bounds on the solution of linear interval systems, with a computable overestimation factor that is frequently quite small. In spite of the provided improvement in this formulation, the two-step approach will result in additional overestimation when evaluating the derived quantities.

It is quite clear that among other factors, the issue of obtaining tight enclosures for the primary variables as well as for the derived quantities is conditioned by IFEM formulation and the methods used for the evaluation of the derived quantities. A new mixed formulation for Interval Finite Element Methods was developed by the authors (Rama Rao, Muhanna and Mullen,, 2011) where the derived quantities of the conventional formulation are treated as dependent variables along with the primary variables. The formulation uses the mixed variational approach based on the Lagrange multiplier method. The system solution provides the primary variables along with the Lagrange multipliers which represent the derived quantities themselves.

The objective of the present work is to utilize this mixed finite element formulation to handle the problem of geometric uncertainty of trusses to obtain sharp bounds on displacement and axial forces. In addition, the structural response of the truss in the simultaneous presence of geometric, load and stiffness uncertainties is evaluated.

3. Formulation

3.1. Geometric uncertainty due to misfit and thermal effects

The present focuses on the issue of geometric uncertainty in truss structures. A *truss* is a structure composed of straight bars connected at their points of intersection by means of pins or hinges (frictionless joints that are not capable of resisting moments). All loadings are assumed to be applied only at these points of intersection. Thus each straight bar is subjected only to axial force, not to shear forces, bending nor twisting moments.

Due to fabrication errors and/or thermal changes certain bars can have improper length. In practice, the bar is forced into its position between two joints by applying some initial extension or compression. Under such a condition, some axial forces are introduced in the bars in the absence of external loads. The solution of such a problem in the absence of uncertainty is well known in the text books of structural engineering. However, based on the engineering practice, the length of the truss bar is introduced as a random value that is equal to the nominal value plus/minus a tolerance. That means, the bar length can have any value between two bounds, namely $L_o - \delta L$ and $L_o + \delta L$, where L_o is the bar nominal length and δL is the given tolerance. In this study we will incorporate uncertainty in the bar length as the range between the lower and upper bounds on the nominal length of the bar.

$$L \in \boldsymbol{L}, \quad \boldsymbol{L} \equiv [\underline{L}, \overline{L}] := \{L \in R \mid \underline{L} \le L \le \overline{L}\} \tag{1}$$

$$\boldsymbol{L} = [L_o - \delta L, L_o + \delta L] \tag{2}$$

The formulation includes two steps and the results of the two steps are then superimposed. Since it is required that all bars have to fit the nominal pre-described geometry, then if a bar is longer than its nominal length it should be compressed to fit into its position between two joints. So when the bar is released it will apply equal and opposite compressive forces on its joints, and if the bar is shorter it will apply a tensile forces. The axial forces developed in all bars due to initial extension/compression or temperature changes can be determined and then can be used to calculate the nodal forces within the finite element context. By doing that, the geometric uncertainty in the bar's length is converted into an equivalent load uncertainty. The interval axial force $\boldsymbol{F}_i^{(e)}$ for a typical bar element i along the element local axes is given by

$$\boldsymbol{F}_i^{(e)} = EA\left(\frac{\boldsymbol{L} - L_0}{L_0}\right) = \frac{EA}{L_0}\left[L_0 - \delta L, L_0 + \delta L\right] = \frac{EA}{L_0}\left[1 - \frac{\delta L}{L_0}, 1 + \frac{\delta L}{L_0}\right] \tag{3a}$$

Where $\boldsymbol{\delta L} = [-\delta L, +\delta L] = [\underline{\delta L}, \overline{\delta L}]$ is the interval deviation from the nominal value of the bar's length, E is the modulus of elasticity, and A is the cross sectional are of the bar. In the case of a temperature change of the interval amount $\boldsymbol{\delta T}$ the interval force will be given by

$$\boldsymbol{F}_i^{(e)} = EA\alpha\boldsymbol{\delta T} \tag{3b}$$

where $\boldsymbol{\delta T} = [-\delta T + \delta T\ ] = [\underline{\delta T}, \overline{\delta T}]$ is the interval of the temperature change, and α is the coefficient of thermal expansion. The combination of fabrication errors and temperature changes can be analyzed using the sum of equivalent forces.

To illustrate how the above mentioned procedure can be applied, let us consider a typical truss bar element as shown in figure 1. According to finite element formulation (Gallagher, 1975; Bathe, 1996; Zienkiewicz and Taylor, 2000), the nodal forces induced by a given bar due fabrication error or temperature change can be determined as

$$\boldsymbol{F_o} = \begin{pmatrix} \boldsymbol{F}_{1x} \\ \boldsymbol{F}_{1y} \\ \boldsymbol{F}_{2x} \\ \boldsymbol{F}_{2y} \end{pmatrix} = EA\frac{\delta \boldsymbol{L}}{L_o}\begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix} \tag{4}$$

Where $\boldsymbol{F}_o$ is the interval vector of nodal forces obtained as a result of the missfitting problem, $c = \cos\varphi$, and $s = \sin\varphi$.

In the absence of external loading the final interval finite element system of equations can be given by

$$\boldsymbol{KU} = \boldsymbol{MF} \tag{5}$$

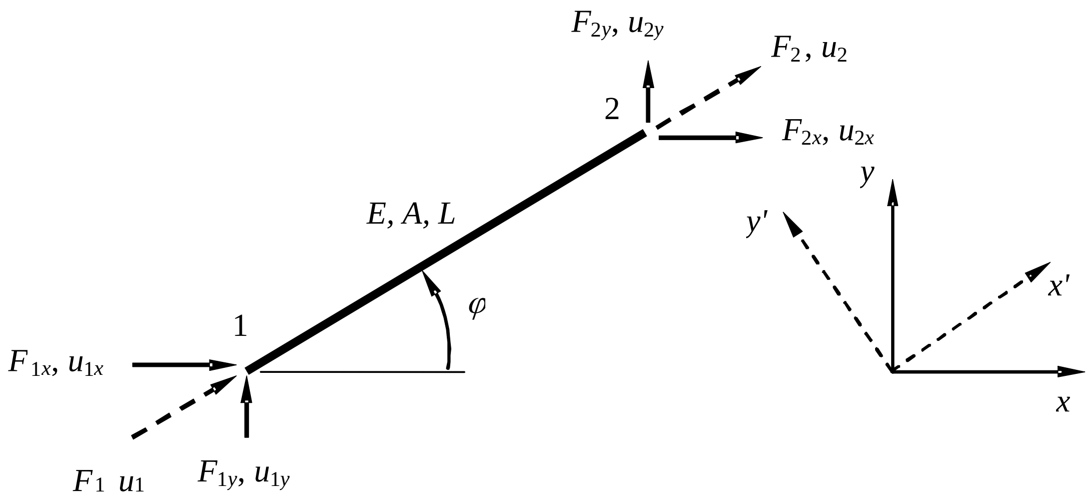

Figure 1. Local (x', y') and global (x, y) coordinate systems for a truss bar element

Where K is the stiffness matrix of the system and $\boldsymbol{U}$ is the vector of interval displacements. Equation (5) is an interval linear system, where only the right hand side is interval and an exact enclosure can be obtained. This enclosure represents the final deformed geometry of the truss due to misfitting.

To obtain the final internal force in each bar, first we need to calculate the internal force in each bar due to the nodal forces $\boldsymbol{P}_o$ using the following equation

$$\boldsymbol{S_i} = K_i L_i \boldsymbol{U} \tag{6}$$

where $\boldsymbol{S}_i$ is the interval force of the i^{th} bar of the truss, K_i is i^{th} element stiffness matrix, and L_i is a Boolean matrix with 1 and 0 entries, and secondly the obtained force should be added to that force given in equation 1 or 2, depending on the case under consideration, i.e. fabrication error or temperature change. In the next section we will introduce some example problems.

However, the above approach followed in the earlier work of the authors (Mullen, Erdolen and Muhanna, 2006) has the following limitations:

a) It can be applied when geometric uncertainty alone is present while no external loads are applied and modulus of elasticity is deterministic.
b) The interval displacements $\boldsymbol{U}$ obtained using Eq. (5) are sharp but the interval axial forces $\mathbf{S}_i$ obtained using Eq. (6) are overestimated. This is because Eq. (6) computes the derived interval axial forces $\mathbf{S}_i$ using interval displacements $\boldsymbol{U}$.
c) This formulation does not handle external loads being applied at the joints.
d) This formulation does not handle uncertainties of load and stiffness.

This brings out the need for a modification of the earlier approach

a) to simultaneously handle geometric, load and stiffness uncertainties

b) to avoid the overestimation of forces and obtain them at the same level of sharpness as the displacements.

Thus the mixed interval finite element formulation (Rama Rao, Muhanna and Mullen, 2011) to handle load and stiffness uncertainties is modified to handle geometric uncertainty as well. The modified formulation is presented in the next sub-section.

3.2. Discrete Structural Models

In steady-state analysis, the variational formulation for a discrete structural model within the context of Finite Element Method (FEM) is given in the following form of the total potential energy functional (Gallagher 1975, Bathe 1996)

$$\Pi = \frac{1}{2}U^T KU - U^T P \tag{7}$$

with the conditions

$$\frac{\partial \Pi}{\partial U_i} = 0 \qquad for\ all\ \ i \tag{8}$$

where Π, K, U, and P are total potential energy, stiffness matrix, displacement vector, and load vector respectively. Assume that we want to impose onto the solution the m linearly independent discrete constraints $CU = V$ where C is a matrix of order $m \times n$. In the Lagrange multiplier method we amend the right-hand side of Eq. (7) to obtain

$$\Pi^* = \frac{1}{2}U^T KU - U^T P + \lambda^T (CU - V) \tag{9}$$

where λ is a vector of m Lagrange multipliers. Invoking the stationarity of Π^*, that is $\delta\Pi^*=0$, we obtain

$$\begin{pmatrix} K & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} U \\ \lambda \end{pmatrix} = \begin{pmatrix} P \\ V \end{pmatrix} \tag{10}$$

The solution of Eq. (10) will provide the values of dependent variable U and λ at the same time.

The present interval formulation is based on the Element-By-Element (EBE) finite element technique developed in the earlier work of authors (Muhanna and Mullen, 2001). In the EBE method, each element has its own set of nodes, but the set of elements is disassembled, so that a node belongs to a single element. A set of additional constraints is introduced to force unknowns associated with coincident nodes to have identical values. Thus, the constraint equation CU=V takes the form

$$CU = 0 \tag{11}$$

where C is the constraint matrix, and equation (10) takes the form:

$$\begin{pmatrix} K & C^T \\ C & 0 \end{pmatrix}\begin{pmatrix} U \\ \lambda \end{pmatrix}=\begin{pmatrix} P \\ 0 \end{pmatrix} \tag{12}$$

3.2.1. *Present Interval Formulation*

The main sources of overestimation in the formulation of IFEM are the multiple occurrences of the same interval variable (*dependency problem*), the width of interval quantities, the problem size, and the problem complexity, in addition to the nature of the used interval solver of the interval linear system of equations. While the present formulation is valid for the FEM models in solid and structural mechanics problems, the truss model will be used here to illustrate the applicability and efficiency of the present formulation of without any loss of generality.

To illustrate the present formulation, let us consider a typical two dimensional truss bar finite element as shown in Figure 1. According to finite element formulation (Bathe, 1996, Gallagher, 1995, Zienkiewicz and Taylor, 2000) the global finite element model of a truss system is given in the following form:

$$KU = P \tag{13}$$

where K is the assembled global stiffness matrix, P is the global load vector, and U is the unknown global displacement vector. Using boldface non-italic font for interval quantities, the interval form of Eq. (13) will be

$$\mathbf{KU} = \mathbf{P} \tag{14}$$

where **K**, **U**, and **P** are the interval global stiffness matrix, interval global displacement vector, and interval global load vector, respectively. The interval solution of Eq. (14) results in a significant overestimation in the system response; a comprehensive discussion can be found in (Muhanna and Mullen, 2001). In addition, internal forces and stresses are quantities of practical interest in design. Usually interval element forces can be obtained as:

$$\mathbf{F_e} = \mathbf{k_e} L_{\mathbf{e}} \mathbf{U} \tag{15}$$

where $\mathbf{F}_e$, $\mathbf{k}_e$, L_e are global interval vector of element forces, global interval element stiffness matrix, and element Boolean matrix, respectively. Once again, an additional overestimation in the values of forces is obtained due to the dependency between **U** and $\mathbf{k}_e$. Frequently, element forces are pursued in local coordinate system that will require the transformation from the global coordinates to the local ones in the form:

$$\mathbf{F}_{e,local} = T_e\, \mathbf{k}_e L_e \mathbf{U} \tag{16}$$

where $\mathbf{F}_{e,\ local}$ and T_e are the local vector of interval element forces and the corresponding transformation matrix, respectively. The transformation procedure will provide an additional overestimation.

The present formulation seeks to reduce overestimation due to coupling in the FEM assembling process, multiple occurrences of interval quantities, transformation, and solving the final system of interval linear equations. In addition this formulation will introduce the derived quantities such as forces and stresses as dependent variables which will be obtained along with displacements when the system is solved.

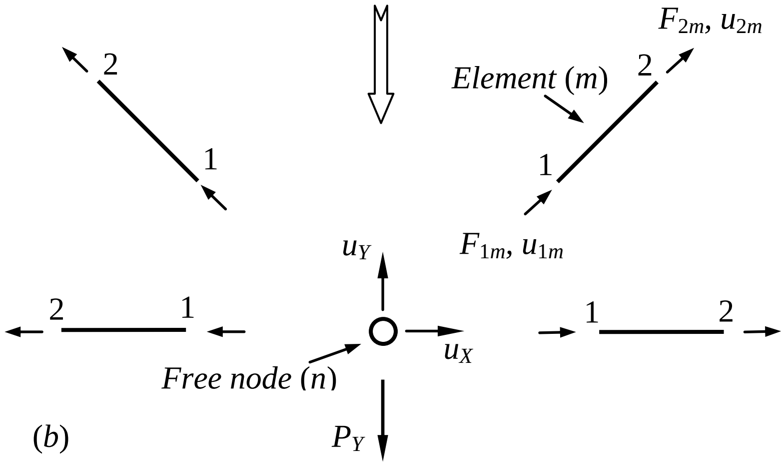

Figure 2. A typical node of a truss problem. (*a*) Conventional formulation. (*b*) Present formulation.

In the conventional formulation of FEM Figure 2 (*a*), after deriving the local elements' stiffness matrices along with the local elements' load vectors the system will be transformed to the global system and assembled based on compatibility requirements resulting in the equilibrium system given by Eq. (14). In the present formulation the following steps are followed:

1. Considering a typical node of the truss system Figure 2 (*a*), elements and nodes are disassembled as in Figure 2 (*b*). The typical node is called a *free node* and is given along with all pertinent variables in the global coordinate system. Displacements are u_X and u_Y and applied forces are P_X and P_Y. The free node displacements are considered as independent of those of the elements.
2. All coinciding elements at the free node along with pertinent variables are given in local coordinate system. For example, element m has the end nodes 1 and 2, the local displacements u_{1m} and u_{2m}, and the local forces F_{1m} and F_{2m}. By doing that, each element is treated as having independent degrees of freedom in its own local coordinate system.
3. The system will be assembled imposing the discrete constraints C_{mi} to ensure the equality between the free node displacements and those of the elements, where i is the number of constraints per element.

This procedure will result in the following system of equations:

$$\begin{pmatrix} \boldsymbol{K} & \mathrm{C}^{\mathrm{T}} \\ \mathrm{C} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{U} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{P} \\ 0 \end{pmatrix} + \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\delta} \\ 0 \end{pmatrix} \tag{17}$$

where **k** is an interval matrix consists of the individual elements' local stiffness and zeros at the bottom corresponding the free nodes' degrees of freedom and have the following structure:

$$[\boldsymbol{K}] = \begin{pmatrix} k_1 & -k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{\mathrm{n}} & -k_{\mathrm{n}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_{\mathrm{n}} & k_{\mathrm{n}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{1\mathrm{X}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0_{1\mathrm{Y}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_{\mathrm{mX}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_{\mathrm{mY}} \end{pmatrix} \tag{18}$$

and

$$\mathbf{k}_i = \frac{\mathbf{E_i A}_i}{L_i} \tag{19}$$

where $\mathbf{E}_i$, $\mathbf{A}_i$, and L_i are the interval modulus of elasticity, the interval cross-sectional area, and the length of each element, respectively.

Matrix C has the dimensions ($k \times l$), where k = number of elements' degrees of freedom ($2 \times$ number of elements in the truss bar element case), and l = total number of the system's degrees of freedom. The entries of the matrix are equality constraints of the following type

$$\mathbf{u}_{1i} + \mathbf{u}_{jX} cos\varphi_i + \mathbf{u}_{jY} sin\varphi_i = 0 \tag{20}$$

Where $\mathbf{u}_{1i}$ is the local displacement of the node 1 that belongs to i^{th} element, $\mathbf{u}_{jX}$ and $\mathbf{u}_{jY}$ are the X and Y global displacements of j^{th} free node coinciding with the 1^{st} node of the i^{th} element. Elements of C^T are shown in Eq. (21).

U is a vector of size $s \times 1$ where s is the number of elements' local degrees of freedom + number of free nodes' global degrees of freedom. The entries of the vector are the interval local displacements of elements followed by interval global displacements of the *free nodes* as shown in Eq. (20).Vector $\boldsymbol{\lambda}$ has the dimension of total number of elements' local degrees of freedom and given in Eq. (21). The entries of the vector are the interval Lagrange multipliers that represent minus the local element forces in this case.

$$C^T = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \cos\varphi_1 & 0 & \cdots \\ \sin\varphi_1 & 0 & \cdots \\ \vdots & \vdots & \cdots \\ 0 & \cos\varphi_1 & \cdots \\ 0 & \sin\varphi_1 & \cdots \end{pmatrix} \quad (22) \qquad \mathbf{U} = \begin{pmatrix} \mathbf{u}_{11} \\ \mathbf{u}_{21} \\ \vdots \\ \mathbf{u}_{1n} \\ \mathbf{u}_{2n} \\ \mathbf{u}_{1X} \\ \mathbf{u}_{1Y} \\ \vdots \\ \mathbf{u}_{mX} \\ \mathbf{u}_{mY} \end{pmatrix} \quad (23) \qquad \boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{\lambda}_{11} \\ \boldsymbol{\lambda}_{21} \\ \vdots \\ \boldsymbol{\lambda}_{1n} \\ \boldsymbol{\lambda}_{2n} \end{pmatrix} \quad (21)$$

Vector **P** is the interval load vector corresponding to the external loads and has the dimension equal to the sum of elements degrees of freedom and the free nodes degrees of freedom. The entries of the vector are given in Eq. (22).

$$\mathbf{P}^T = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \mathbf{P}_{1X} & \mathbf{P}_{1Y} & \cdots & \mathbf{P}_{mX} & \mathbf{P}_{mY} \end{pmatrix} \quad (22)$$

The misfit forces acting along the element axes can be represented for the entire structure using M matrix approach as

$$\boldsymbol{F} = [M]\{\boldsymbol{\delta}\} \quad (23)$$

where M matrix is deterministic and contains contributions of the nominal axial stiffness values of elements and $\boldsymbol{\delta}$ contains the normalized interval multipliers due to misfit, representing geometric uncertainty.

It may be noted that the interval multiplier for the element 'i' can be obtained from Eq. (3a) as

$$\boldsymbol{\delta_i} = \left[1 - \frac{\delta L}{L_0}, 1 + \frac{\delta L}{L_0}\right] \quad (24)$$

Overestimation is avoided by

a) applying the element forces are applied at the EBE nodes of the structure along local axes and
b) keeping element forces separate throughout the course of the solution

The accuracy of the system solution depends mainly on the structure of Eq. (18) and on the nature of the used solver. The solution of the interval system (17) provides the enclosures of the values of dependent variables which are the interval displacements **U** and interval element forces **λ**. An iterative solver is discussed in the next section.

2.5. ITERATIVE ENCLOSURES

The best known method for obtaining very sharp enclosures of interval linear system of equations that have the structure introduced in Eq. (14) is the iterative method developed in the work of Neumaier and Pownuk, (2007) that introduces the system of interval linear equations in the form:

$$\left[[\mathrm{K}_0]+[\mathrm{B}][\boldsymbol{D}][\mathrm{A}]\right]\{u\}=\{\mathrm{a}\}+[\mathrm{F}]\{\boldsymbol{b}\} \tag{26}$$

with interval quantities in **D** and **b** only.

Considering Eq. (26), we take K_0 as a zero matrix, and express the global interval stiffness matrix ***K*** of the structure as

$$[\boldsymbol{K}]=\left[A^T\right][\boldsymbol{D}][A] \tag{27}$$

and further, considering $\{\mathrm{a}\}$ as vector of zeros, and **F** as **M** and **b** as **δ** as given in Eq. (23), we obtain

$$\left[A^T\right][\boldsymbol{D}][A]\{\boldsymbol{U}\}=[\mathrm{M}]\{\boldsymbol{\delta}\} \tag{28}$$

Thus, the current formulation results in the interval linear system of equations given in (17) which can be introduced in the same structure of Eq. (26), and the interval enclosure for **U** suggested by Neumaier and Pownuk, (2007) can be obtained.

However, it is to be noted that in Neumaier's work only excellent enclosures of the interval displacements are obtained. However for the derived quantities such as forces he suggested an improved enclosure which results in additional significant overestimation using the mixed finite element formulation developed earlier by the authors (Rama Rao, Muhanna and Mullen, 2011). The current formulation allows obtaining the interval displacement **U** and the accompanied interval derived quantities **λ** with the same accuracy. Example problems are solved in the following section to illustrate the excellent accuracy of the developed method.

4. Example problem

Two example problems, viz. a six bar truss and a thirteen bar truss are chosen as example problems to illustrate the present approach and also to demonstrate its ability to obtain sharp bounds to the displacements and forces even in the presence of simultaneous presence of geometric, load and material uncertainties.

A one-bay truss of six elements, shown in Figure 3 is chosen as the first example problem. The deterministic value of Young's modulus of each element is E_i=200 GPa, i=1,2,…,6, while the cross sectional area is 0.01m^2. The modulus of elasticity of each element is assumed to vary independently. The truss is subjected to a nominal load of P=100 kN , applied at the top left joint corresponding to node 3.

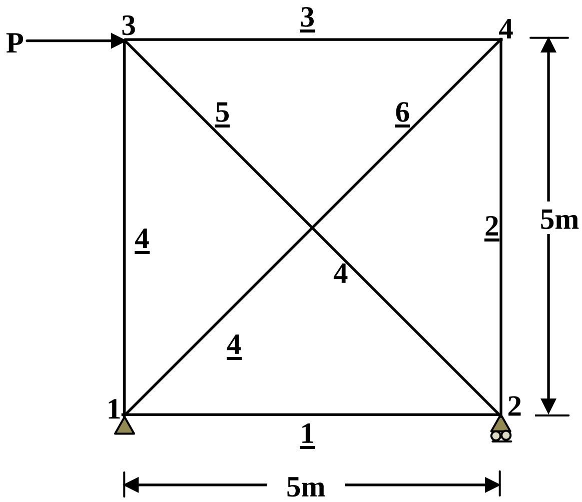

Figure 3. Six bar truss.

Three cases are as outlined below are considered and results are computed using the present interval approach and also combinatorial approach.

A. The displacements and forces of the truss due to fabrication error (geometric uncertainty) alone are computed. External load is not considered in this case. The same fabrication error of 0.2 percent (±0.1 percent variation about nominal value of length) for all members is assumed in both example problems. This corresponds to a misfitting strain of $\frac{\delta L}{L_0}$ =0.001. **(Case A).**

B. The truss is subjected to external loads. Both geometric uncertainty (due to fabrication error and load uncertainty are considered. The load uncertainty considered is 10 percent corresponding to a variation of ±5 percent about the nominal value of the load. Thus the interval load considered is P=[95,105] kN. The displacements and axial forces computed correspond to the simultaneous presence of geometric and load uncertainties **(Case B).**

C. The truss is subjected to external loads. Stiffness uncertainty arising out of uncertainty of modulus of elasticity is considered along with uncertainty of geometry and loading. Thus the interval value of modulus of elasticity is [199,201] GPa. The displacements and axial forces computed correspond to the simultaneous presence of geometric, load and stiffness uncertainties **(Case C).**

Table 1 shows the computed values of selected displacements (horizontal displacement U_3, vertical displacement V_3 at node 3 and horizontal displacement U_4 at node 4) using the two approaches mentioned above for a geometric uncertainty of 0.2 percent (±0.1% from the mean value of length of element) for the first case mentioned above . Table 2 shows the corresponding values of selected axial forces N_2, N_3, N_6 computed for members 2,3 and 6. It is observed from Tables 1 and 2 that the displacements and forces obtained using the present approach at 1 percent uncertainty provides an exact enclosure to the corresponding displacements and forces obtained using the combinatorial approach.

Table 1 Six bar truss - displacements for 0.2 percent geometric uncertainty alone						
	$U_3 \times 10^{-3}$		$V_3 \times 10^{-3}$		$U_4 \times 10^{-3}$	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-3.41421	3.41421	-1.49999	1.49999	-3.20710	3.20710
Present approach	-3.41421	3.41421	-1.50000	1.50000	-3.20710	3.20710
Error %(bounds)	0.00	0.00	0.00	0.00	0.00	0.00

Table 2 Six bar truss – axial forces for 0.2 percent geometric uncertainty alone						
	N_2 (kN)		N_3 (kN)		N_6 (kN)	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-200.0	200.0	-141.42	141.42	-141.42	141.42
Present approach	-200.0	200.0	-141.42	141.42	-141.42	141.42
Error %(bounds)	0.0	0.0	0.0	0.0	0.0	0.0

Table 3 shows the computed values of selected displacements (horizontal displacement U_3, vertical displacement V_3 at node 3 and horizontal displacement U_4 at node 4) using the two approaches mentioned above for a geometric uncertainty of 0.2 percent (±0.1% from the mean value of length of element) and 10 percent load uncertainty, for the second case mentioned above . Table 5 shows the corresponding values of selected axial forces N_2, N_3, N_6 computed for members 2,3 and 6. It is observed from Tables 3 and 4 that the displacements and forces obtained using the present approach at 1 percent uncertainty provides an exact enclosure to the corresponding displacements and forces obtained using the combinatorial approach.

Table 3 Six bar truss - displacements for 0.2 percent geometric uncertainty and 10 percent load uncertainty						
	$U_3 \times 10^{-3}$		$V_3 \times 10^{-5}$		$U_4 \times 10^{-5}$	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-2.26746	4.68167	-1.26250	1.76249	-2.29785	4.21206
Present approach	-2.26746	4.68167	-1.26250	1.76250	-2.29785	4.21206
Error %(bounds)	0.00	0.00	0.00	0.0005	0.00	0.00

Table 4 Six bar truss – axial forces for 0.2 percent geometric uncertainty and 10 percent load uncertainty						
	N_2 (kN)		N_3 (kN)		N_6 (kN)	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-274.24	132.82	-193.92	93.92	-93.92	193.92
Present approach	-274.24	132.82	-193.92	93.92	-93.92	193.92
Error %(bounds)	0.0	0.0	0.0	0.0	0.0	0.0

Table 5 shows the computed values of selected displacements (horizontal displacement U_3, vertical displacement V_3 at node 3 and horizontal displacement U_4 at node 4) using the two approaches mentioned above for a geometric uncertainty of 0.2 percent (±0.1% from the mean value of length of element) and 10 percent load uncertainty along with a stiffness uncertainty (of modulus of elasticity) of 1 percent, for the third case mentioned above. Table 6 shows the corresponding values of selected axial forces N_2, N_3, N_6 computed for members 2, 3 and 6. It is observed from Tables 5 and 6 that the displacements and forces obtained using the present approach at 1 percent uncertainty provides a very sharp enclosure to the corresponding displacements and forces obtained using the combinatorial approach. It is also observed from Tables 5 and 6 that forces and displacements are obtained with the same level of sharpness. Thus it is observed that the present approach provides guaranteed bounds on the combinatorial approach in all the three cases in the presence of geometric, load and stiffness uncertainties. This brings out the efficiency of the present approach.

Table 5 Six bar truss - displacements for 0.2 percent geometric uncertainty,10 percent load uncertainty and 1 percent uncertainty in modulus of Elasticity (E)

	$U_3 \times 10^{-3}$		$V_3 \times 10^{-5}$		$U_4 \times 10^{-5}$	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-2.28453	4.69874	-1.26999	1.76999	-2.31389	4.22810
Present approach	-2.35837	4.77259	-1.29639	1.79639	-2.38431	4.29853
Error %(bounds)	3.23	1.57	2.07	1.49	3.04	1.66

Table 6 Six bar truss – forces for 0.2 percent geometric uncertainty,10 percent load uncertainty and 1 percent uncertainty in modulus of Elasticity (E)

	N_2 (kN)		N_3 (kN)		N_6 (kN)	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-275.91	134.45	-194.96	94.94	-94.94	194.96
Present approach	-279.57	138.15	-197.89	97.89	-97.68	197.68
Error %(bounds)	1.32	2.74	1.50	3.10	2.88	1.39

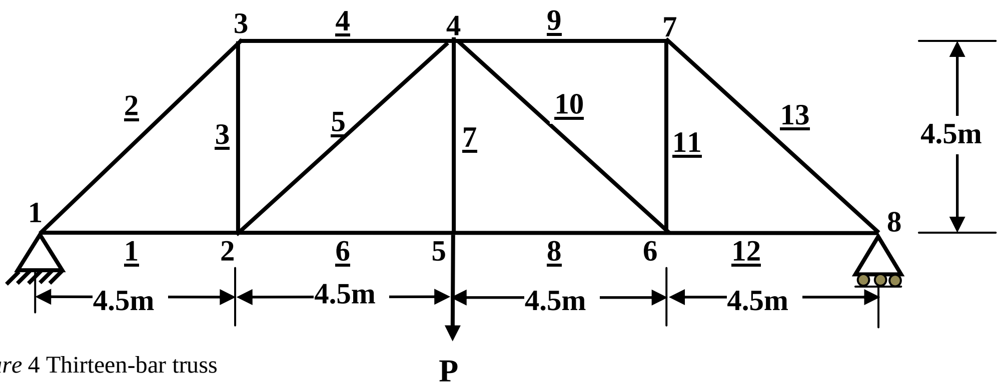

Figure 4 Thirteen-bar truss

The thirteen bar truss shown in Figure 4 is chosen as the second example problem. The truss is subjected to a vertical point load of P=200 kN acting downwards applied at the joint 5. Cross section areas of elements 1,2,3,13,14 and 15 are 10.0×10^{-4} m^2 while for the rest of the elements is the cross sectional area is 6.0×10^{-4} m^2. The deterministic value of Young's modulus of each element is E_i=200 GPa, i =1, 2 ,…15. The following cases are considered for computation of displacements and axial forces using interval approach and combinatorial approach.

D. Modulus of elasticity is deterministic and the geometric uncertainty for all members is 0.2 percent while no external load is applied.
E. The interval load applied is [190,210] kN corresponding to a load uncertainty of 10 percent. All elements have a deterministic value of stiffness of 200 GPa. Geometric uncertainty is 0.2 percent for all members
F. The interval load applied is [190,210] kN. Stiffness uncertainty of elements 4, 7, 9 and 10 is 1 percent (corresponding to an interval modulus of elasticity of [199,201] GPa while the remaining

elements have a deterministic value of stiffness of 200 GPa. Geometric uncertainty is 0.2 percent for all members

Table 7 shows the computed values of selected displacements (horizontal displacement U_3, vertical displacement V_5 at node 5 and vertical displacement V_6 at node 6) using the two approaches mentioned above for a geometric uncertainty of 0.2 percent and 10 percent load uncertainty, for the case D mentioned above . It is observed from Table 7 that the displacements and forces obtained using the present approach at 1 percent uncertainty provides an exact enclosure to the corresponding displacements and forces obtained using the combinatorial approach.

Table 7 Thirteen bar truss - displacements for 0.2 percent geometric uncertainty alone

	$U_3 \times 10^{-3}$		$V_5 \times 10^{-3}$		$V_6 \times 10^{-3}$	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-4.91421	4.91421	-8.82842	8.82842	-5.62132	5.62132
Present approach	-4.91421	4.91421	-8.82842	8.82842	-6.12132	6.12132
Error %(bounds)	0.0	0.0	0.0	0.0	0.0	0.0

Table 8 shows the computed values of selected displacements (horizontal displacement U_3, vertical displacement V_5 at node 5 and vertical displacement V_6 at node 6) using the two approaches mentioned above for the case E mentioned above . Table 9 shows the corresponding values of selected axial forces N_2, N_5, N_6 computed for members 2, 5 and 6 for case E. It is observed from Tables 8 and 9 that the displacements and forces obtained using the present approach at 1 percent uncertainty provides a very sharp enclosure to the corresponding displacements and forces obtained using the combinatorial approach.

Table 8 Thirteen bar truss - displacements for 0. 2 percent geometric uncertainty and 10 percent load uncertainty

	$U_3 \times 10^{-2}$		$V_5 \times 10^{-3}$		$V_6 \times 10^{-2}$	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	0.79107	1.90892	-5.89350	-3.65061	-2.93409	-1.48869
Present approach	0.79107	1.90892	-5.89350	-3.65061	-2.93409	-1.48869
Error %(bounds)	0.0	0.0	0.0	0.0	0.0	0.0

Table 9 Thirteen bar truss – axial forces for 0.2 percent geometric uncertainty and 10 percent load uncertainty

	N_2 (kN)		N_5 (kN)		N_6 (kN)	
	Lower	Upper	Lower	Upper	Lower	Upper
Combinatorial approach	-148.49	-134.35	-148.49	-134.35	190.00	210.00
Present approach	-148.49	-134.35	-148.49	-134.35	190.00	210.00
Error %(bounds)	0.0	0.0	0.0	0.0	0.0	0.0

5. Conclusions

A method for the analysis of structures and mechanical components with geometric, load and stiffness uncertainties is presented. This method is based on the use of a mixed finite element formulation that ensures that forces have the same level of sharpness as displacements. The need for maintaining parametric relationships within the interval formulations is ensured by an element-by-element approach to the formulations. Exact enclosure on the deformed geometry is obtained. In addition, the use of mixed finite element formulation ensures that axial forces are obtained at the same level of sharpness as displacements in the simultaneous presence of geometric, stiffness and load uncertainties. Example calculations are presented that show the sharpness of the interval calculations.

6. Acknowledgements

The first author would like to gratefully acknowledge the help received via international travel funding by the Technical Education Quality Improvement Program (TEQIP- Phase II) of Government of India and its sanction by the administration of Vasavi College of Engineering, Hyderabad, India.

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