Reliability Analysis of Structures with Interval Uncertainties under Stochastic Excitations

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Abstract: In this paper, an approach able to evaluate approximate explicit expressions of interval response statistics and structural reliability of randomly excited linear structures with uncertain parameters is addressed. The excitation is modeled as a stationary Gaussian random process. Uncertainty in structural parameters is handled by applying the interval model. Under the assumption of independent up-crossings of a specified threshold, a procedure for the analytical derivation of interval reliability sensitivity is presented. The key idea is to perform stochastic analysis in the frequency domain by applying the *improved interval analysis* in conjunction with the so-called *Interval Rational Series Expansion* (*IRSE*). A wind-excited truss structure with interval stiffness properties is analyzed to show the effectiveness of the proposed procedure.

Keywords: Reliability function; Stochastic excitation; Interval uncertainties; Upper bound and lower bound.

1. Introduction

Uncertainties affecting structural parameters, such as geometrical and mechanical properties, are commonly described within a probabilistic framework as random variables or random fields with assigned probability density function (*PDF*). However, the credibility of traditional probabilistic reliability methods relies on the availability of sufficient data to describe accurately the probabilistic distribution of the uncertain variables, especially in the tails. Indeed, reliability estimates are very sensitive to small variations of the assumed probabilistic models (Ben-Haim, 1994; Elishakoff, 1995). In the last decades, several non-probabilistic approaches (Elishakoff and Ohsaki, 2010) have been developed for handling uncertainties described by fragmentary or incomplete data.

In the non-probabilistic formulation of reliability, the attention was focused on the definition of non-probabilistic reliability measures alternative to the traditional ones (see e.g. Penmetsa and Grandhi, 2002; Luo et al., 2009; Kang et al., 2011; Beer et al., 2013; Hurtado, 2013; Jiang et al., 2013).

Studies on reliability analysis of randomly excited structures with uncertain parameters (Gupta and Manohar, 2006; Chaudhuri and Chakraborty, 2006; Taflanidis, 2010) have been carried out mainly within the probabilistic framework focusing on the estimation of the statistics of the failure probability (or reliability). Much less attention has been devoted to reliability analysis of structures under stochastic excitation especially in presence of uncertain parameters with incomplete information.

To fill this gap, the present contribution deals with the reliability evaluation of linear structural systems with uncertain-but-bounded parameters subjected to stationary Gaussian random excitation. Due to the uncertainty affecting structural parameters, the reliability turns out to be an interval function and structural performance ranges between lower and upper bounds. The main objective of this study is to develop a procedure for deriving approximate explicit expressions of the interval reliability function.

The failure is assumed here to occur as the random process modeling the response quantity of interest (e.g. displacement, stress or strain at a critical point) firstly exceeds a safe domain within a specified time interval $[0,T]$. The proposed procedure is developed under the assumption that consecutive crossings of a specified threshold are statistically independent events so as to constitute approximately a Poisson process (Lutes and Sarkani, 1997; Muscolino and Palmeri, 2005; Li and Chen, 2009). As known, in this case the reliability evaluation involves the zero- and second-order spectral moments of a selected stationary response process.

In this paper, the interval spectral moments of the stochastic response process and the corresponding interval reliability function are evaluated in approximate closed-form by applying an approach recently proposed by the authors (Muscolino and Sofi, 2013; Muscolino et al., 2014a,b, in press). Basically, this approach relies on the use of the so-called *Interval Rational Series Expansion* (*IRSE*) (Muscolino and Sofi, 2012a, 2013) in conjunction with the *improved interval analysis* (Muscolino and Sofi, 2012b, 2013), introduced to limit the overestimation due to the *dependency phenomenon* (Muhanna and Mullen, 2001; Moens and Vandepitte, 2005) affecting the *"ordinary"* or *classical interval analysis* (Moore, 1966). For small deviation amplitudes of the interval parameters the *first-order interval Taylor series expansion* (Muscolino and Sofi, 2011) is adopted to obtain accurate estimates of the upper and lower bounds of the interval reliability.

2. Improved Interval Analysis: Basic Definitions

The interval model is a widely used non-probabilistic approach to handle uncertainties occurring in engineering problems. This model, stemming from the interval analysis (Moore, 1966), turns out to be very useful when only the range of variability of the uncertain parameters is available. In this section, the basic notations and definitions of the so-called *improved interval analysis*, recently proposed (Muscolino and Sofi, 2012b, 2013) to overcome the main limitations of the *"ordinary"* or *classical interval analysis* in structural engineering applications are introduced.

Denoting by IR the set of all closed real interval numbers, let $\boldsymbol{\alpha}^I \equiv [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}] \in IR^r$ be a bounded set-interval vector of real numbers, such that $\underline{\boldsymbol{\alpha}} \leq \boldsymbol{\alpha} \leq \overline{\boldsymbol{\alpha}}$. In the following the apex *I* denotes interval variable while the symbols $\underline{\boldsymbol{\alpha}}$ and $\overline{\boldsymbol{\alpha}}$ denote the lower bound (*LB*) and upper bound (*UB*) vectors. Since the real numbers α_i, collected into the vector $\boldsymbol{\alpha}$, are bounded by intervals, all mathematical derivations involving α_i should be performed by means of the *classical interval analysis* (Moore, 1966). Unfortunately, the *classical interval analysis* suffers from the so-called *dependency phenomenon* (Muhanna and Mullen, 2001; Moens and Vandepitte, 2005) which often leads to an overestimation of the interval solution width that could be catastrophic from an engineering point of view.

To limit the effects of the dependency phenomenon, the so-called *generalized interval analysis* (Hansen, 1975) and the *affine arithmetic* (Comba and Stolfi, 1993; Stolfi and De Figueiredo, 2003) have been introduced in the literature. In these formulations, each intermediate result is represented by a linear function with a small remainder interval (Nedialkov et al., 2004). Following the philosophy of the *affine arithmetic*, within the framework of structural analysis, the so-called *improved interval analysis* has been proposed (Muscolino and Sofi, 2012b, 2013). The key feature of this approach is the introduction of the *extra symmetric unitary interval* (*EUI*), $\hat{e}_i^I \triangleq [-1,+1]$, $(i=1,2,...,r)$, defined in such a way that the following properties hold:

$$\hat{e}_i^I - \hat{e}_i^I = 0; \quad \hat{e}_i^I \times \hat{e}_i^I = \left(\hat{e}_i^I\right)^2 = [1,1]; \quad \hat{e}_i^I / \hat{e}_i^I = [1,1]$$
$$\hat{e}_i^I \times \hat{e}_j^I = [-1,+1], \quad i \neq j; \quad x_i \hat{e}_i^I \pm y_i \hat{e}_i^I = (x_i \pm y_i)\hat{e}_i^I; \tag{1a-f}$$
$$x_i \hat{e}_i^I \times y_i \hat{e}_i^I = x_i y_i \left(\hat{e}_i^I\right)^2 = x_i y_i [1,1].$$

In these equations, $[1,1]=1$ is the so-called unitary *thin interval*. It is useful to remember that a thin interval occurs when $\underline{x} = \overline{x}$ and it is defined as $x^I \triangleq [\underline{x}, \underline{x}]$, so that $x \in \mathrm{R}$. Notice that the *EUI* is different from the *classical unitary symmetric interval* $e^I \triangleq [-1,+1]$ which does not satisfy Eqs. (1a-f) (Muscolino et al., 2013).

Then, introducing the midpoint value (or mean), $\alpha_{0,i}$, and the deviation amplitude (or radius), $\Delta\alpha_i$, of the *i*-th real interval variable α_i^I:

$$\alpha_{0,i} = \frac{1}{2}\left(\underline{\alpha}_i + \overline{\alpha}_i\right); \quad \Delta\alpha_i = \frac{1}{2}\left(\overline{\alpha}_i - \underline{\alpha}_i\right), \tag{2a,b}$$

the *improved interval analysis* assumes the following *affine form* definition for the interval parameter α_i^I:

$$\alpha_i^I = \alpha_{0,i} + \Delta\alpha_i \hat{e}_i^I, \qquad (i=1,2,\dots,r). \tag{3}$$

In the sequel, $\boldsymbol{\alpha}_0$ and $\Delta\boldsymbol{\alpha}$ will denote the vectors listing the midpoint values (or mean values) and the deviation amplitudes (or radii), $\alpha_{0,i}$ and $\Delta\alpha_i$, respectively, of the interval parameters α_i^I, $(i=1,2,...,r)$, collected into the vector $\boldsymbol{\alpha}^I$. Furthermore, in the framework of interval symbolism, a generic interval-valued function f and a generic interval-valued matrix function $\mathbf{A}$ of the interval vector $\boldsymbol{\alpha}^I$ will be denoted in equivalent form, respectively, as:

$$f^I \equiv f(\boldsymbol{\alpha}^I) \quad \Leftrightarrow \quad f(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}];$$
$$\mathbf{A}^I \equiv \mathbf{A}(\boldsymbol{\alpha}^I) \quad \Leftrightarrow \quad \mathbf{A}(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\alpha}}]. \tag{4a,b}$$

3. Equations Governing the Problem

Let us consider a quiescent *n*-DOF linear structure with uncertain-but-bounded parameters subjected to a stationary multi-correlated Gaussian stochastic process $\mathbf{f}(t)$. The equations of motion can be cast in the following form:

$$\mathbf{M}_0\ddot{\mathbf{u}}(\boldsymbol{\alpha},t)+\mathbf{C}(\boldsymbol{\alpha})\dot{\mathbf{u}}(\boldsymbol{\alpha},t)+\mathbf{K}(\boldsymbol{\alpha})\mathbf{u}(\boldsymbol{\alpha},t)=\mathbf{f}(t), \quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\bar{\boldsymbol{\alpha}}\right] \tag{5}$$

where $\mathbf{M}_0$ is the $n\times n$ mass matrix, herein assumed deterministic; $\mathbf{C}(\boldsymbol{\alpha})$ and $\mathbf{K}(\boldsymbol{\alpha})$ are the $n\times n$ damping and stiffness matrices of the structure which depend on the dimensionless interval parameters collected into the vector $\boldsymbol{\alpha}\in\boldsymbol{\alpha}^I$ of order *r*; $\mathbf{u}(\boldsymbol{\alpha},t)$ is the stationary Gaussian vector process of displacements; and a dot over a variable denotes differentiation with respect to time t. The Rayleigh model is herein adopted for the interval damping matrix, i.e.:

$$\mathbf{C}(\boldsymbol{\alpha})=c_0\mathbf{M}_0+c_1\mathbf{K}(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\bar{\boldsymbol{\alpha}}\right] \tag{6}$$

where c_0 and c_1 are the Rayleigh damping constants having units s^{-1} and s, respectively.

In structural engineering, the dimensionless fluctuations of the uncertain-but-bounded parameters around their nominal values can be reasonably modelled as symmetric intervals, i.e. $\alpha_i^I\in\left[\underline{\alpha}_i,\bar{\alpha}_i\right]$ with $\bar{\alpha}_i=-\underline{\alpha}_i$. Under this assumption, Eq. (3) reduces to:

$$\alpha_i^I=\Delta\alpha_i\hat{e}_i^I \tag{7}$$

being $\alpha_{0,i}=0$ and $\Delta\alpha_i=-\underline{\alpha}_i=\bar{\alpha}_i$. Furthermore, to assure physically meaningful values of the uncertain structural properties, the deviation amplitudes $\Delta\alpha_i$ should satisfy the conditions $\left|\Delta\alpha_i\right|<1$, with the symbol $\left|\bullet\right|$ denoting absolute value.

The external load vector $\mathbf{f}(t)$ in Eq.(5) is modelled here as a stationary multi-correlated Gaussian random process. It is fully characterized, from a probabilistic point of view, by the mean-value vector, $\boldsymbol{\mu}_{\mathbf{f}}=\mathrm{E}\langle\mathbf{f}(t)\rangle$, and the correlation function matrix, $\mathbf{R}_{\mathbf{ff}}(t_2-t_1)=\mathbf{R}_{\mathbf{ff}}(\tau)=\mathrm{E}\langle\mathbf{f}(t_1)\mathbf{f}^T(t_2)\rangle-\boldsymbol{\mu}_{\mathbf{f}}\,\boldsymbol{\mu}_{\mathbf{f}}^T$, with $\mathrm{E}\langle\bullet\rangle$ denoting the stochastic average operator and the apex *T* transpose matrix.

As well-known, the response of a linear structural system subjected to a Gaussian random process is Gaussian too and can be characterized in the so-called frequency domain by the mean-value vector and the *PSD* function matrix (see e.g. Lutes and Sarkani, 1997; Li and Chen, 2009). In particular, the stationary stochastic Gaussian interval response process of structures with uncertain-but-bounded parameters is completely characterized in the frequency domain once the interval mean-value vector, $\boldsymbol{\mu}_{\mathbf{u}}(\boldsymbol{\alpha})$, and the interval *PSD* function matrix, $\mathbf{G}_{\mathbf{uu}}(\boldsymbol{\alpha},\omega)$, $\boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\bar{\boldsymbol{\alpha}}\right]$ are known.

The interval stiffness matrix $\mathbf{K}(\boldsymbol{\alpha})\equiv\mathbf{K}^I$, $\boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\bar{\boldsymbol{\alpha}}\right]$, of a linearly-elastic structure can be expressed as a linear function of the uncertain parameters. Based on this observation and following the interval formalism introduced above, the $n\times n$ order interval stiffness and damping matrices can be expressed as linear functions of the uncertain properties, i.e.:

$$\mathbf{K}^I=\mathbf{K}_0+\sum_{i=1}^{r}\mathbf{K}_i\,\Delta\alpha_i\,\hat{e}_i^I;$$
$$\mathbf{C}^I=\mathbf{C}_0+c_1\sum_{i=1}^{r}\mathbf{K}_i\,\Delta\alpha_i\,\hat{e}_i^I \tag{8a,b}$$

where

$$\mathbf{K}_0 = \mathbf{K}(\boldsymbol{\alpha})\big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}; \quad \mathbf{K}_i = \frac{\partial \mathbf{K}(\boldsymbol{\alpha})}{\partial \alpha_i}\bigg|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}; \qquad (9a\text{-}c)$$
$$\mathbf{C}_0 = c_0\mathbf{M}_0 + c_1\mathbf{K}_0.$$

In the previous equations, $\mathbf{K}_0$ and $\mathbf{C}_0$ denote the stiffness and damping matrices of the nominal structural system, which are positive definite symmetric matrices of order $n\times n$; $\mathbf{K}_i$ are positive semi-definite symmetric matrices of order $n\times n$; $\Delta\alpha_i$ is the dimensionless deviation amplitude of the *i*-th uncertain parameter satisfying the condition $|\Delta\alpha_i|<1$.

The interval mean-value of the response process governed by Eq. (5), where the input has mean-value $\boldsymbol{\mu}_{\mathbf{f}} = \mathrm{E}\langle \mathbf{f}(t)\rangle$, can be determined once the inverse of the *interval stiffness matrix* (see Eq. (8a)) is evaluated, that is:

$$\boldsymbol{\mu}_{\mathbf{u}}(\boldsymbol{\alpha}) = \mathrm{E}\langle \mathbf{u}(\boldsymbol{\alpha},t)\rangle = \mathbf{K}^{-1}(\boldsymbol{\alpha})\boldsymbol{\mu}_{\mathbf{f}}, \qquad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}]. \qquad (10)$$

On the other hand, the interval response *PSD* function matrix can be determined by means of the following relationship (Li and Chen, 2009):

$$\mathbf{G}_{\mathbf{uu}}(\boldsymbol{\alpha},\omega) = \mathbf{H}^*(\boldsymbol{\alpha},\omega)\mathbf{G}_{\tilde{\mathbf{X}}_{\mathbf{f}}\tilde{\mathbf{X}}_{\mathbf{f}}}(\omega)\mathbf{H}^T(\boldsymbol{\alpha},\omega), \quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}] \qquad (11)$$

where the asterisk means complex conjugate; $\mathbf{G}_{\tilde{\mathbf{X}}_{\mathbf{f}}\tilde{\mathbf{X}}_{\mathbf{f}}}(\omega)$ is the *PSD* function matrix of the zero-mean stationary Gaussian stochastic process, $\tilde{\mathbf{X}}_{\mathbf{f}}(t)$, describing the random fluctuation of the input process, i.e. $\mathbf{f}(t) = \boldsymbol{\mu}_{\mathbf{f}} + \tilde{\mathbf{X}}_{\mathbf{f}}(t)$, such that $\mathbf{R}_{\mathbf{ff}}(\tau) \equiv \mathbf{R}_{\tilde{\mathbf{X}}_{\mathbf{f}}\tilde{\mathbf{X}}_{\mathbf{f}}}(\tau)$; $\mathbf{H}(\boldsymbol{\alpha},\omega)$ is the interval *frequency response function (FRF) matrix* (also referred to as *transfer function matrix*), defined as follows:

$$\mathbf{H}(\boldsymbol{\alpha},\omega) = \left[\mathbf{H}_0^{-1}(\omega) + \mathbf{P}(\boldsymbol{\alpha},\omega)\right]^{-1} = \left[\mathbf{I}_n + \mathbf{H}_0(\omega)\mathbf{P}(\boldsymbol{\alpha},\omega)\right]^{-1}\mathbf{H}_0(\omega), \quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I = [\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}]. \qquad (12)$$

In the previous equation, $\mathbf{I}_n$ denotes the identity matrix of order *n* and $\mathbf{H}_0(\omega)$ is the *FRF* matrix of the nominal structural system, given by:

$$\mathbf{H}_0(\omega) = \left[-\omega^2\mathbf{M}_0 + \mathrm{j}\omega\mathbf{C}_0 + \mathbf{K}_0\right]^{-1} \qquad (13)$$

where $\mathrm{j}=\sqrt{-1}$ denotes the imaginary unit. Furthermore, according to the symbolism introduced in Eq. (4 b), the matrix $\mathbf{P}(\boldsymbol{\alpha},\omega)\equiv\mathbf{P}^I(\omega)$ is defined as:

$$\mathbf{P}^I(\omega) = p(\omega)\sum_{i=1}^{r}\mathbf{K}_i\Delta\alpha_i\,\hat{e}_i^I; \quad p(\omega) = (1+\mathrm{j}\omega c_1). \qquad (14a,b)$$

$\mathbf{P}^I(\omega)$ is an interval complex matrix of order $n\times n$ accounting for the fluctuations of the structural parameters affecting the stiffness and damping matrices (see Eqs. (8a,b)). Notice that for classically damped structural systems, the nominal *FRF* matrix can be conveniently evaluated as:

$$\mathbf{H}_0(\omega) = \boldsymbol{\Phi}_0\,\mathbf{H}_{0,\mathrm{m}}(\omega)\boldsymbol{\Phi}_0^T \qquad (15)$$

where $\boldsymbol{\Phi}_0$ is the nominal modal matrix of order $n\times s$ $(s\le n)$ collecting the first *s* eigenvectors normalized with respect to the mass matrix $\mathbf{M}_0$. This matrix is obtained as solution of the eigenproblem pertaining to the nominal structural system, i.e. for $\boldsymbol{\alpha}=\boldsymbol{\alpha}_0$. In Eq.(15), $\mathbf{H}_{0,\mathrm{m}}(\omega)$ is the nominal modal *transfer function matrix* that, for classically damped structural systems, is a diagonal matrix given by:

$$\mathbf{H}_{0,\mathrm{m}}(\omega)=\left[-\omega^2\mathbf{I}_s+\mathrm{j}\,\omega\,\boldsymbol{\Xi}_0+\boldsymbol{\Omega}_0^2\right]^{-1} \tag{16}$$

where $\boldsymbol{\Omega}_0^2=\boldsymbol{\Phi}_0^T\,\mathbf{K}_0\,\boldsymbol{\Phi}_0$ is the spectral matrix of the nominal structure say a diagonal matrix listing the squares of the natural circular frequencies of the structure for $\boldsymbol{\alpha}=\boldsymbol{\alpha}_0$; $\boldsymbol{\Xi}_0=\boldsymbol{\Phi}_0^T\mathbf{C}_0\boldsymbol{\Phi}_0$ denotes the generalised nominal damping matrix, which for the Rayleigh model of damping assumed here is a diagonal matrix.

Once, the interval *PSD* function matrix, $\mathbf{G}_{\mathbf{uu}}(\boldsymbol{\alpha},\omega)$, $\boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}]$, is known, the matrix collecting the interval spectral moments of order ℓ of the random response, useful for structural reliability evaluation, can be computed as (Vanmarcke, 1972):

$$\boldsymbol{\Lambda}_{\ell,\mathbf{uu}}(\boldsymbol{\alpha})=2\int_0^\infty \omega^\ell\,\mathbf{G}_{\mathbf{uu}}(\boldsymbol{\alpha},\omega)\,\mathrm{d}\omega,\quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right];\quad \ell=0,1,2. \tag{17}$$

As known, the generic response quantity of practical interest, $Y(\boldsymbol{\alpha},t)$, can be determined from the displacement vector $\mathbf{u}(\boldsymbol{\alpha},t)$ by means of the following relationship:

$$Y(\boldsymbol{\alpha},t)=\mathbf{q}^T\,\mathbf{u}(\boldsymbol{\alpha},t)\Rightarrow Y(\boldsymbol{\alpha},\omega)=\mathbf{q}^T\,\mathbf{U}(\boldsymbol{\alpha},\omega)=\mathbf{q}^T\,\mathbf{H}(\boldsymbol{\alpha},\omega)\mathbf{F}(\omega),\qquad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right] \tag{18}$$

where $\mathbf{q}$ is a vector collecting the combination coefficients relating the response process $Y(\boldsymbol{\alpha},t)$ to $\mathbf{u}(\boldsymbol{\alpha},t)$; $Y(\boldsymbol{\alpha},\omega)$ is the Fourier Transform of $Y(\boldsymbol{\alpha},t)$; $\mathbf{U}(\boldsymbol{\alpha},\omega)$ and $\mathbf{F}(\omega)$ are the Fourier Transforms of the displacement and forcing vectors, $\mathbf{u}(\boldsymbol{\alpha},t)$ and $\mathbf{f}(t)$, respectively; $\mathbf{H}(\boldsymbol{\alpha},\omega)$ is the interval *FRF* matrix given by Eq. (12). Depending on the selected response quantity, $Y(\boldsymbol{\alpha},t)$, the vector $\mathbf{q}$ may depend on the uncertain parameters; such dependency is here not considered for the sake of simplicity.

The interval stationary Gaussian random process, $Y(\boldsymbol{\alpha},t)=\mu_Y(\boldsymbol{\alpha})+\tilde{Y}(\boldsymbol{\alpha},t)$, is completely characterized by the interval mean-value, $\mu_Y(\boldsymbol{\alpha})$, and the interval *PSD* function, $G_{YY}(\boldsymbol{\alpha},\omega)\equiv G_{\tilde{Y}\tilde{Y}}(\boldsymbol{\alpha},\omega)$ of the zero-mean random process $\tilde{Y}(\boldsymbol{\alpha},t)$ with $\boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right]$.

The interval mean-value, $\mu_Y(\boldsymbol{\alpha},t)$, can be evaluated taking the stochastic average of both sides of Eq. (18), i.e.:

$$\mu_Y(\boldsymbol{\alpha})=\mathrm{E}\left\langle Y(\boldsymbol{\alpha},t)\right\rangle=\mathbf{q}^T\,\boldsymbol{\mu}_{\mathbf{u}}(\boldsymbol{\alpha}),\qquad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right] \tag{19}$$

where $\boldsymbol{\mu}_{\mathbf{u}}(\boldsymbol{\alpha})$ is the mean-value of the displacement vector $\mathbf{u}(\boldsymbol{\alpha},t)$ given by Eq. (10).

Based on Eq. (18), the interval *PSD* function of the zero-mean response process $\tilde{Y}(\boldsymbol{\alpha},t)$ reads:

$$G_{\tilde{Y}\tilde{Y}}(\boldsymbol{\alpha},\omega)\equiv G_{YY}(\boldsymbol{\alpha},\omega)=\mathbf{q}^T\mathbf{G}_{\mathbf{uu}}\left(\boldsymbol{\alpha},\omega\right)\mathbf{q}=\mathbf{q}^T\,\mathbf{H}^*(\boldsymbol{\alpha},\omega)\mathbf{G}_{\tilde{\mathbf{X}}_\mathbf{f}\tilde{\mathbf{X}}_\mathbf{f}}(\omega)\mathbf{H}^T(\boldsymbol{\alpha},\omega)\mathbf{q}. \tag{20}$$

Finally, the spectral moments of order ℓ of the random process $\tilde{Y}(\boldsymbol{\alpha},t)$, coincident with the spectral moments of $Y(\boldsymbol{\alpha},t)$, can be evaluated from Eq.(20) as follows:

$$\lambda_{\ell,\tilde{Y}}(\boldsymbol{\alpha})\equiv\lambda_{\ell,Y}(\boldsymbol{\alpha})=2\int_0^\infty \omega^\ell\,G_{YY}(\boldsymbol{\alpha},\omega)\,\mathrm{d}\omega=\mathbf{q}^T\boldsymbol{\Lambda}_{\ell,\mathbf{uu}}(\boldsymbol{\alpha})\mathbf{q},\quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right];\quad \ell=0,1,2. \tag{21}$$

4. Interval Stationary Stochastic Response

As outlined in the previous section, the interval stationary Gaussian random response process $\mathbf{u}(\boldsymbol{\alpha}^I, t)$ is completely characterized in the frequency domain by the interval mean-value vector, $\boldsymbol{\mu}_{\mathbf{u}}(\boldsymbol{\alpha}^I)$, and the interval *PSD* function matrix, $\mathbf{G}_{\mathbf{uu}}(\boldsymbol{\alpha}^I, \omega)$, whose evaluation basically requires: *i*) to compute the inverse of the parametric interval stiffness matrix, $\mathbf{K}^{-1}(\boldsymbol{\alpha}^I)$ (see Eq.(10)); *ii*) to evaluate the interval *FRF* matrix, $\mathbf{H}(\boldsymbol{\alpha}^I, \omega)$ (see Eq. (11)) which involves the inversion of a parametric frequency-dependent matrix (see Eq.(12)). By applying the *IRSE*, recently, the authors (Muscolino and Sofi, 2012a, 2013; Muscolino et al., 2014a) proposed a procedure for evaluating these matrices and the associated statistics of the interval stationary response in approximate explicit form. The starting point to derive the *IRSE* is the decomposition of the $n \times n$ matrix $\mathbf{K}_i$ in Eq. (9b) as sum of rank-one matrices. Without loss of generality, the particular case in which the matrix $\mathbf{K}^I$ of the structural system in Eq.(8a) exhibits rank-one modifications is here examined. This circumstance occurs for truss structures, shear-type frames or discretized structures with modifications in axial components only and so on. In these cases, the matrices $\mathbf{K}_i$ have rank one and the stiffness matrix of the structural system with *r* uncertain parameters in Eq. (8a) can be expressed as follows (Impollonia and Muscolino, 2011):

$$\mathbf{K}^I = \mathbf{K}_0 + \sum_{i=1}^{r} \mathbf{K}_i \Delta\alpha_i \, \hat{e}_i^I = \mathbf{K}_0 + \sum_{i=1}^{r} \mathbf{v}_i \mathbf{v}_i^T \Delta\alpha_i \, \hat{e}_i^I \tag{22}$$

where $\mathbf{v}_i$ is a vector of order *n* such that the dyadic product $\mathbf{K}_i = \mathbf{v}_i \mathbf{v}_i^T$ gives a change of rank one. It follows that the deviation with respect to the nominal value is expressed as superposition of *r* rank-one matrices.

Retaining only first-order terms, the *IRSE* yields the following approximate explicit expression of the inverse of the interval stiffness matrix (Impollonia and Muscolino, 2011; Muscolino and Sofi, 2013):

$$\left(\mathbf{K}^I\right)^{-1} = \left[\mathbf{K}_0 + \sum_{i=1}^{r} \mathbf{v}_i \mathbf{v}_i^T \Delta\alpha_i \, \hat{e}_i^I\right]^{-1} \approx \mathbf{K}_0^{-1} - \sum_{i=1}^{r} \frac{\Delta\alpha_i \, \hat{e}_i^I}{1 + \Delta\alpha_i \, \hat{e}_i^I d_i} \, \mathbf{D}_i \tag{23}$$

where

$$d_i = \mathbf{v}_i^T \mathbf{K}_0^{-1} \mathbf{v}_i; \quad \mathbf{D}_i = \mathbf{K}_0^{-1} \mathbf{v}_i \mathbf{v}_i^T \mathbf{K}_0^{-1}. \tag{24a,b}$$

Notice that Eq. (23) holds if and only if the conditions $\left|\Delta\alpha_i \, d_i\right| < 1$ are satisfied. Obviously, the accuracy of Eq.(23) depends on the magnitude of the fluctuations $\Delta\alpha_i$. Higher-order terms can be included to handle larger uncertainties (Muscolino and Sofi, 2012a).

By applying the *IRSE* in Eq. (23) in conjunction with the *improved interval analysis* (Muscolino and Sofi, 2012b, 2013), and adopting the formalism introduced in Eq. (4b), the approximate interval mean-value response vector can be expressed as:

$$\boldsymbol{\mu}_{\mathbf{u}}^I \equiv \boldsymbol{\mu}_{\mathbf{u}}(\boldsymbol{\alpha}^I) = \left(\mathbf{K}^I\right)^{-1} \boldsymbol{\mu}_{\mathbf{f}} = \mathrm{mid}\left\{\boldsymbol{\mu}_{\mathbf{u}}(\boldsymbol{\alpha}^I)\right\} + \mathrm{dev}\left\{\boldsymbol{\mu}_{\mathbf{u}}(\boldsymbol{\alpha}^I)\right\} \tag{25}$$

where

$$\operatorname{mid}\left\{\boldsymbol{\mu}_{\mathbf{u}}^{I}\right\}=\left[\mathbf{K}_0^{-1}+\sum_{i=1}^{r}\tilde{a}_{0,i}\mathbf{D}_i\right]\boldsymbol{\mu}_{\mathbf{f}};\quad \operatorname{dev}\left\{\boldsymbol{\mu}_{\mathbf{u}}^{I}\right\}=\left[\sum_{i=1}^{r}\Delta\tilde{a}_i\,\hat{e}_i^I\mathbf{D}_i\right]\boldsymbol{\mu}_{\mathbf{f}} \tag{26a,b}$$

are the midpoint and deviation vectors which are defined in explicit form. Indeed, in Eqs. (26a,b), $\tilde{a}_{0,i}$ and $\Delta\tilde{a}_i$ are functions of the deviation amplitude $\Delta\alpha_i$, given by (Muscolino and Impollonia, 2011):

$$\tilde{a}_{0,i}=\frac{\Delta\alpha_i^2 d_i}{1-\left(\Delta\alpha_i d_i\right)^2};\quad \Delta\tilde{a}_i=\frac{\Delta\alpha_i}{1-\left(\Delta\alpha_i d_i\right)^2} \tag{27a,b}$$

where the argument $\Delta\alpha_i$ is omitted for conciseness.

Based on Eq. (25), the mean-value vector, $\boldsymbol{\mu}_{\mathbf{u}}(\boldsymbol{\alpha})$, turns out to be an interval vector with midpoint given by Eq. (26a) and width or deviation amplitude defined as follows:

$$\Delta\boldsymbol{\mu}_{\mathbf{u}}(\boldsymbol{\alpha})=\sum_{i=1}^{r}\Delta\tilde{a}_i\left|\mathbf{D}_i\,\boldsymbol{\mu}_{\mathbf{f}}\right| \tag{28}$$

with the symbol $|\bullet|$ denoting absolute value component wise. Then, the *LB* and *UB* of the interval mean-value vector, $\boldsymbol{\mu}_{\mathbf{u}}(\boldsymbol{\alpha})$, can be readily obtained as:

$$\underline{\boldsymbol{\mu}}_{\mathbf{u}}(\boldsymbol{\alpha})=\operatorname{mid}\left\{\boldsymbol{\mu}_{\mathbf{u}}^{I}\right\}-\Delta\boldsymbol{\mu}_{\mathbf{u}}(\boldsymbol{\alpha});\quad \overline{\boldsymbol{\mu}}_{\mathbf{u}}(\boldsymbol{\alpha})=\operatorname{mid}\left\{\boldsymbol{\mu}_{\mathbf{u}}^{I}\right\}+\Delta\boldsymbol{\mu}_{\mathbf{u}}(\boldsymbol{\alpha}). \tag{29a,b}$$

Upon substitution of the decomposition $\mathbf{K}_i=\mathbf{v}_i\mathbf{v}_i^T$ into Eq.(14a), the interval *FRF* matrix, $\mathbf{H}(\boldsymbol{\alpha}^I,\omega)$ (see Eq. (12)), of the structural system with interval parameters takes the following form:

$$\mathbf{H}^I(\omega)=\left[\mathbf{H}_0^{-1}(\omega)+p(\omega)\sum_{i=1}^{r}\mathbf{v}_i\mathbf{v}_i^T\Delta\alpha_i\,\hat{e}_i^I\right]^{-1}. \tag{30}$$

The decomposition (22) of the stiffness matrix thus allows to express also the deviation with respect to the inverse of the nominal *FRF* matrix, $\mathbf{H}_0^{-1}(\omega)$, as sum of r modifications of rank one. Then, by applying the *IRSE* (Muscolino and Sofi, 2013, 2012a; Muscolino et al., 2014a), an approximate explicit expression of the interval *FRF* matrix is obtained, i.e.:

$$\mathbf{H}^I(\omega)=\left[\mathbf{H}_0^{-1}(\omega)+\sum_{i=1}^{r}p(\omega)\Delta\alpha_i\,\hat{e}_i^I\,\mathbf{v}_i\mathbf{v}_i^T\right]^{-1}\approx\mathbf{H}_0(\omega)-\sum_{i=1}^{r}\frac{p(\omega)\Delta\alpha_i\,\hat{e}_i^I}{1+p(\omega)\Delta\alpha_i\,\hat{e}_i^I\,b_i(\omega)}\mathbf{B}_i(\omega) \tag{31}$$

where the following complex frequency dependent quantities appear:

$$b_i(\omega)=\mathbf{v}_i^T\mathbf{H}_0(\omega)\mathbf{v}_i;\quad \mathbf{B}_i(\omega)=\mathbf{H}_0(\omega)\mathbf{v}_i\mathbf{v}_i^T\mathbf{H}_0(\omega). \tag{32a,b}$$

In the previous equations, $\mathbf{H}_0(\omega)$ is the *FRF* matrix of the nominal structural system which can be evaluated by means of Eq.(15). Equation (31) holds if and only if the conditions $\left\|p(\omega)\Delta\alpha_i\,b_i(\omega)\right\|<1$ are satisfied, where the symbol $\|\bullet\|$ means modulus of $\bullet$.

Alternatively, by applying the *improved interval analysis* (Muscolino and Sofi, 2013; Muscolino et al. 2014a) and adopting the symbolism introduced in Eq. (4b), the approximate interval *FRF* matrix in Eq.(31), $\mathbf{H}^I(\omega)$, can be rewritten as follows:

$$\mathbf{H}(\boldsymbol{\alpha},\omega)=\mathbf{H}_{\text{mid}}(\omega)+\mathbf{H}_{\text{dev}}(\boldsymbol{\alpha},\omega),\quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right] \tag{33}$$

where:

$$\mathbf{H}_{\mathrm{mid}}(\boldsymbol{\alpha},\omega)=\mathrm{mid}\left\{\mathbf{H}^{I}(\omega)\right\}=\mathbf{H}_{0}(\omega)+\sum_{i=1}^{r}a_{0,i}(\omega)\,\mathbf{B}_{i}(\omega);$$
$$\mathbf{H}_{\mathrm{dev}}^{I}(\omega)=\mathrm{dev}\left\{\mathbf{H}^{I}(\omega)\right\}=\sum_{i=1}^{r}\Delta a_{i}(\omega)\,\mathbf{B}_{i}(\omega)\,\hat{e}_{i}^{I} \qquad (34a,b)$$

are two complex function matrices whose elements are the midpoint and the deviation of the elements of $\mathbf{H}^{I}(\omega)$, respectively. In Eqs. (34a,b), $a_{0,i}(\omega)$ and $\Delta a_{i}(\omega)$ denote complex functions depending on both ω and $\Delta\alpha_{i}$, given respectively, by:

$$a_{0,i}(\omega)=\frac{\left[p(\omega)\Delta\alpha_{i}\right]^{2}b_{i}(\omega)}{1-\left[p(\omega)\Delta\alpha_{i}\,b_{i}(\omega)\right]^{2}};\quad \Delta a_{i}(\omega)=\frac{p(\omega)\Delta\alpha_{i}}{1-\left[p(\omega)\Delta\alpha_{i}\,b_{i}(\omega)\right]^{2}} \qquad (35a,b)$$

where the argument $\Delta\alpha_{i}$ is omitted for the sake of conciseness.

Substituting the interval transfer function matrix, $\mathbf{H}(\boldsymbol{\alpha},\omega)$, written in the form (33) into Eq.(11), the interval *PSD* function matrix of the structural response can be expressed as:

$$\begin{aligned}\mathbf{G}_{\mathbf{uu}}(\boldsymbol{\alpha},\omega)&\equiv\mathbf{G}_{\mathbf{uu}}^{I}(\omega)=\mathbf{H}^{*}(\boldsymbol{\alpha},\omega)\,\mathbf{G}_{\tilde{\mathbf{X}}_{\mathbf{f}}\tilde{\mathbf{X}}_{\mathbf{f}}}(\omega)\,\mathbf{H}^{T}(\boldsymbol{\alpha},\omega)\\&=\mathrm{mid}\left\{\mathbf{G}_{\mathbf{uu}}(\boldsymbol{\alpha},\omega)\right\}+\mathrm{dev}\left\{\hat{\mathbf{G}}_{\mathbf{uu}}(\boldsymbol{\alpha},\omega)\right\},\qquad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^{I}=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right]\end{aligned} \qquad (36)$$

where

$$\mathrm{mid}\left\{\mathbf{G}_{\mathbf{uu}}^{I}(\omega)\right\}=\mathbf{H}_{\mathrm{mid}}^{*}(\boldsymbol{\alpha},\omega)\,\mathbf{G}_{\tilde{\mathbf{X}}_{\mathbf{f}}\tilde{\mathbf{X}}_{\mathbf{f}}}(\omega)\,\mathbf{H}_{\mathrm{mid}}^{T}(\boldsymbol{\alpha},\omega);$$
$$\mathrm{dev}\left\{\hat{\mathbf{G}}_{\mathbf{uu}}^{I}(\omega)\right\}=\mathbf{H}_{\mathrm{mid}}^{*}(\boldsymbol{\alpha},\omega)\,\mathbf{G}_{\tilde{\mathbf{X}}_{\mathbf{f}}\tilde{\mathbf{X}}_{\mathbf{f}}}(\omega)\left(\mathbf{H}_{\mathrm{dev}}^{I}(\omega)\right)^{T}+\left(\mathbf{H}_{\mathrm{dev}}^{I}(\omega)\right)^{*}\mathbf{G}_{\tilde{\mathbf{X}}_{\mathbf{f}}\tilde{\mathbf{X}}_{\mathbf{f}}}(\omega)\,\mathbf{H}_{\mathrm{mid}}^{T}(\boldsymbol{\alpha},\omega) \qquad (37a,b)$$

denote the midpoint and deviation matrices, respectively. The over hat in Eq. (37b) means that an approximate expression of the deviation matrix is assumed. Specifically, in order to simplify interval computations, terms associated with powers of $\Delta\alpha_{i}$ greater than one are neglected (Muscolino et al., 2014a,b).

Finally, upon substitution of Eq. (36) into Eq. (17), approximate analytical expressions of the interval spectral moments of order ℓ of the random response are obtained:

$$\boldsymbol{\Lambda}_{\ell,\mathbf{uu}}(\boldsymbol{\alpha})=\mathrm{mid}\left\{\boldsymbol{\Lambda}_{\ell,\mathbf{uu}}(\boldsymbol{\alpha},\omega)\right\}+\mathrm{dev}\left\{\boldsymbol{\Lambda}_{\ell,\mathbf{uu}}(\boldsymbol{\alpha},\omega)\right\},\quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^{I}=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right];\quad \ell=0,1,2 \qquad (38)$$

where

$$\mathrm{mid}\left\{\boldsymbol{\Lambda}_{\ell,\mathbf{uu}}\right\}=2\int_{0}^{\infty}\omega^{\ell}\,\mathrm{mid}\left\{\mathbf{G}_{\mathbf{uu}}^{I}(\omega)\right\}\mathrm{d}\omega;\quad \mathrm{dev}\left\{\boldsymbol{\Lambda}_{\ell,\mathbf{uu}}\right\}=2\int_{0}^{\infty}\omega^{\ell}\,\mathrm{dev}\left\{\hat{\mathbf{G}}_{\mathbf{uu}}^{I}(\omega)\right\}\mathrm{d}\omega. \qquad (39a,b)$$

It is worth emphasizing that Eqs. (25), (36) and (38) provide the explicit relationships between the interval statistics of the displacement vector $\mathbf{u}(\boldsymbol{\alpha},t)$ and the radius $\Delta\alpha_{i}$ of the input parameters useful for the analytical evaluation of the corresponding interval sensitivities. Furthermore, analogous expressions can be derived for the statistics $\mu_{Y}(\boldsymbol{\alpha})$, $G_{\tilde{Y}\tilde{Y}}(\boldsymbol{\alpha},\omega)\equiv G_{YY}(\boldsymbol{\alpha},\omega)$ and $\lambda_{\ell,\tilde{Y}}(\boldsymbol{\alpha})\equiv\lambda_{\ell,Y}(\boldsymbol{\alpha})$ of the generic response process $Y(\boldsymbol{\alpha},t)$ (see Eq. (18)), by substituting Eqs. (25), (36) and (38) into Eqs. (19)-(21), respectively.

5. Explicit Interval Reliability Function

Structural systems are conceived and designed to survive natural actions, such as earthquake ground motion, wind gusts, sea wave motion, etc. Since such excitations are commonly modeled as stochastic processes, the structural safety needs to be evaluated within a probabilistic framework. Among the available models of failure, the simplest one, which is also the most widely used in practical analyses, is based on the assumption that a structure fails as soon as the response at a critical location exceeds a prescribed safe domain for the first time. Accordingly, the probability of failure, for structural systems subjected to stochastic excitations, coincides with the *first passage probability*, i.e. the probability that the *extreme value* random process, $Y_{\max}(\boldsymbol{\alpha},T)$, for the generic structural response process of interest, $Y(\boldsymbol{\alpha},t)$, (e.g. displacement, strain or stress at a critical point), firstly exceeds the safety bounds within a specified time interval $[0,T]$.

For a structure with uncertain-but-bounded parameters, the *extreme value* random process, over a time interval $[0,T]$, is mathematically defined as:

$$Y_{\max}(\boldsymbol{\alpha},T)=\max_{0\le t\le T}\left|Y(\boldsymbol{\alpha},t)\right|,\quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right] \tag{40}$$

where the symbol $|\bullet|$ denotes absolute value.

Since the *cumulative distribution function* (*CDF*) $L_{Y_{\max}}(\boldsymbol{\alpha},b,T)$ of the *extreme value* random process, $Y_{\max}(\boldsymbol{\alpha},T)$, also called *reliability function*, formally coincides with the *CDF* of the random process $\tilde{Y}_{\max}(\boldsymbol{\alpha},T)=\max_{0\le t\le T}\left|\tilde{Y}(\boldsymbol{\alpha},t)\right|$, where $\tilde{Y}(\boldsymbol{\alpha},T)$ denotes the zero-mean stationary stochastic process describing the random fluctuation of $Y(\boldsymbol{\alpha},t)=\mu_Y(\boldsymbol{\alpha})+\tilde{Y}(\boldsymbol{\alpha},t)$, it is very useful to evaluate the function, $L_{\tilde{Y}_{\max}}(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha}),T)$. This function, which represents the probability that $\tilde{Y}_{\max}(\boldsymbol{\alpha},T)$ is equal to or less than the barrier level $\tilde{b}(\boldsymbol{\alpha})=b-\boldsymbol{\mu}_Y(\boldsymbol{\alpha})$ within the time interval $[0,T]$, is commonly expressed as (Vanmarcke, 1975):

$$L_{\tilde{Y}_{\max}}\left(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha}),T\right)=\mathcal{P}\left[\tilde{Y}_{\max}(\boldsymbol{\alpha},T)\le\tilde{b}(\boldsymbol{\alpha})\right]=P_{0,\tilde{Y}}(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha}))\exp\left[-T\,\eta_{\tilde{Y}}(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha}))\right],\quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right] \tag{41}$$

where $P_{0,\tilde{Y}}(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha}))=\mathcal{P}[\tilde{Y}_{\max}(\boldsymbol{\alpha},0)\le\tilde{b}(\boldsymbol{\alpha})]$ denotes the initial probability, that is the probability of not exceeding the level $\tilde{b}(\boldsymbol{\alpha})$ at time $t=0$; $\eta_{\tilde{Y}}(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha}))$ is the so-called *hazard function* (or *intensity function*, or *limiting decay rate*). The latter provides the conditional occurrence rate of up-crossings at time *t* of the level $\tilde{b}(\boldsymbol{\alpha})$ by the random process $\left|\tilde{Y}(\boldsymbol{\alpha},t)\right|$, given that this level has not been up-crossed prior to *t*. Since the *hazard function* is not available in exact form, different approximate expressions have been proposed in the literature (see e.g. Lutes and Sarkani, 1997; Li and Chen, 2005; Muscolino and Palmeri, 2005).

It is recognized that, if the failure level is high enough, then the classical Rice's formula, based on the Poisson assumption of independent up-crossings (Rice, 1950), is applicable. In this case, the interval *hazard function* $\eta_{\tilde{Y}}\left(\boldsymbol{\alpha}^I,\tilde{b}(\boldsymbol{\alpha}^I)\right)$, for Gaussian processes with sufficiently large mean-value, $\mu_Y(\boldsymbol{\alpha}^I)>0$, can be assumed, with good accuracy, as coincident with the mean up-crossing rate of a single barrier $\tilde{b}(\boldsymbol{\alpha}^I)=b-\mu_Y(\boldsymbol{\alpha}^I)$ (Vanmarcke, 1975). Then, the interval *CDF*, $L_{\tilde{Y}_{\max}}(\boldsymbol{\alpha}^I,\tilde{b}(\boldsymbol{\alpha}^I),T)$, can be expressed as:

$$L_{\tilde{Y}_{\max}}\left(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha}),T\right)=\mathcal{P}\left[\tilde{Y}_{\max}(\boldsymbol{\alpha},T)\leq\tilde{b}(\boldsymbol{\alpha})\right]$$
$$\approx P_{0,\tilde{Y}}\left(\boldsymbol{\alpha},\tilde{b}(\boldsymbol{\alpha})\right)\exp\left[-T\,\nu_{\tilde{Y}}^{+}(\boldsymbol{\alpha})\exp\left(-\frac{\tilde{b}^2(\boldsymbol{\alpha})}{2\lambda_{0,\tilde{Y}}(\boldsymbol{\alpha})}\right)\right],\quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right] \tag{42}$$

where

$$\nu_{\tilde{Y}}^{+}(\boldsymbol{\alpha})\equiv\nu_{Y}^{+}(\boldsymbol{\alpha})=\frac{1}{2\,\pi}\sqrt{\frac{\lambda_{2,Y}(\boldsymbol{\alpha})}{\lambda_{0,Y}(\boldsymbol{\alpha})}};\quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right] \tag{43}$$

is the mean up-crossing rate at level $\mu_Y(\boldsymbol{\alpha}^I)>0$; $P_{0,\tilde{Y}}\left(\boldsymbol{\alpha}^I,\tilde{b}(\boldsymbol{\alpha}^I)\right)$ represents the initial interval probability, herein assumed equal to unity; $\lambda_{0,Y}(\boldsymbol{\alpha}^I)\equiv\lambda_{0,\tilde{Y}}(\boldsymbol{\alpha}^I)$ and $\lambda_{2,Y}(\boldsymbol{\alpha}^I)\equiv\lambda_{2,\tilde{Y}}(\boldsymbol{\alpha}^I)$ are the interval spectral moments of zero- and second-order, respectively, of the generic structural response process of interest $Y(\boldsymbol{\alpha}^I,t)$, given by Eq. (21).

Let $X_{\max}(\boldsymbol{\alpha}^I,T)=\tilde{Y}_{\max}(\boldsymbol{\alpha}^I,T)/\sqrt{\lambda_{0,\tilde{Y}}(\boldsymbol{\alpha}^I)}$ denote the largest dimensionless extremum process, often referred to in the literature as *peak factor process* during the observation time *T*. Introducing the dimensionless interval variable $\beta_{0,Y}(\boldsymbol{\alpha}^I)=\left(b-\mu_Y(\boldsymbol{\alpha}^I)\right)/\sqrt{\lambda_{0,Y}(\boldsymbol{\alpha}^I)}=\tilde{b}(\boldsymbol{\alpha}^I)/\sqrt{\lambda_{0,Y}(\boldsymbol{\alpha}^I)}$, the interval *CDF*, given in Eq.(42), for unit initial probability, can be rewritten as:

$$L_{X_{\max}}\left(\boldsymbol{\alpha},\beta_{0,Y}(\boldsymbol{\alpha}),T\right)=\mathcal{P}\left[X_{\max}(\boldsymbol{\alpha},T)\leq\beta_{0,Y}(\boldsymbol{\alpha})\right]$$
$$\approx\exp\left[-T\,\nu_{Y}^{+}(\boldsymbol{\alpha})\exp\left(-\frac{\beta_{0,Y}^2(\boldsymbol{\alpha})}{2}\right)\right],\quad \boldsymbol{\alpha}\in\boldsymbol{\alpha}^I=\left[\underline{\boldsymbol{\alpha}},\overline{\boldsymbol{\alpha}}\right]. \tag{44}$$

It is worth emphasizing that the interval *CDF* of the *peak factor* in Eq. (44) can be expressed in approximate explicit form by substituting the analytical expressions (21) of the interval spectral moments, $\lambda_{0,Y}(\boldsymbol{\alpha}^I)\equiv\lambda_{0,\tilde{Y}}(\boldsymbol{\alpha}^I)$ and $\lambda_{2,Y}(\boldsymbol{\alpha}^I)\equiv\lambda_{2,\tilde{Y}}(\boldsymbol{\alpha}^I)$, evaluated by applying the *IRSE*.

In order to derive a closed-form expression of the *CDF* of the *peak factor*, under the assumption of slight deviation amplitudes of the uncertain parameters, i.e. $|\Delta\alpha_i|\ll 1$, the interval *CDF*, $L_{X_{\max}}^I\left(\beta_{0,Y},T\right)$ can be approximated by applying the *first-order interval Taylor series expansion* (Alefeld and Herzberger, 1983):

$$L_{X_{\max}}^I\left(\beta_{0,Y}^{(0)},T\right)=L_{X_{\max}}^{(0)}\left(\beta_{0,Y}^{(0)},T\right)+\sum_{i=1}^{r}s_{L_{X_{\max}},i}^I\left(\beta_{0,Y}^{(0)},T\right)\Delta\alpha_i$$
$$=L_{X_{\max}}^{(0)}\left(\beta_{0,Y}^{(0)},T\right)+\sum_{i=1}^{r}\Delta s_{L_{X_{\max}},i}\left(\beta_{0,Y}^{(0)},T\right)\Delta\alpha_i\,\hat{e}_i^I \tag{45}$$

where $L_{X_{\max}}^{(0)}\left(\beta_{0,Y}^{(0)},T\right)$ is the nominal or midpoint *CDF* of the *peak factor*, defined by:

$$L_{X_{\max}}^{(0)}\left(\beta_{0,Y}^{(0)},T\right)\equiv L_{X_{\max}}\left(\boldsymbol{\alpha}^{I},\beta_{0,Y}(\boldsymbol{\alpha}^{I}),T\right)\Big|_{\Delta\boldsymbol{\alpha}=\mathbf{0}}=\exp\left[-T\,\nu_{Y,0}^{+}\,\exp\left(-\frac{\left(\beta_{0,Y}^{(0)}\right)^{2}}{2}\right)\right] \quad (46)$$

where $\nu_{Y,0}^{+}\equiv\nu_{Y}^{+}(\boldsymbol{\alpha})\big|_{\Delta\boldsymbol{\alpha}=\mathbf{0}}=(1/2\,\pi)\sqrt{\lambda_{2,Y}^{(0)}/\lambda_{0,Y}^{(0)}}$ is the mean up-crossing rate pertaining to the structure with nominal values of the uncertain parameters and $\beta_{0,Y}^{(0)}=\beta_{0,Y}(\boldsymbol{\alpha}^{I})\big|_{\Delta\boldsymbol{\alpha}=\mathbf{0}}=\left(b-\mu_{Y,0}\right)\Big/\sqrt{\lambda_{0,Y}^{(0)}}$ is the nominal value of the dimensionless interval variable $\beta_{0,Y}(\boldsymbol{\alpha}^{I})$, $\mu_{Y,0}$ being the nominal mean-value of the random process.

In Eq. (45), $s_{L_{X_{\max}},i}^{I}\left(\beta_{0,Y}^{(0)},T\right)$ and $\Delta s_{L_{X_{\max}},i}\left(\beta_{0,Y}^{(0)},T\right)$ denote the *i*-th *peak factor reliability sensitivity* and the corresponding deviation amplitude given, respectively, by:

$$\begin{aligned}s_{L_{X_{\max}},i}^{I}\left(\beta_{0,Y}^{(0)},T\right)&=\frac{\partial L_{X_{\max}}^{I}\left(\boldsymbol{\alpha},\beta_{0,Y}(\boldsymbol{\alpha}),T\right)}{\partial\Delta\alpha_{i}}\bigg|_{\Delta\boldsymbol{\alpha}=\mathbf{0}}\\&=C(\beta_{0,Y}^{(0)},T)\left\{2\beta_{0,Y}^{(0)}\sqrt{\lambda_{0,Y}^{(0)}}\,s_{\mu_{Y},i}^{I}+\left[\left(\beta_{0,Y}^{(0)}\right)^{2}-1\right]s_{\lambda_{0,Y},i}^{I}+\frac{\lambda_{0,Y}^{(0)}}{\lambda_{2,Y}^{(0)}}\,s_{\lambda_{2,Y},i}^{I}\right\}\end{aligned} \quad (47)$$

and

$$\Delta s_{L_{X_{\max}},i}\left(\beta_{0,Y}^{(0)},T\right)=C\left(\beta_{0,Y}^{(0)},T\right)\left\{2\,\beta_{0,Y}^{(0)}\sqrt{\lambda_{0,Y}^{(0)}}\,\Delta s_{\mu_{Y},i}+\left[\left(\beta_{0,Y}^{(0)}\right)^{2}-1\right]\Delta s_{\lambda_{0,Y},i}+\frac{\lambda_{0,Y}^{(0)}}{\lambda_{2,Y}^{(0)}}\,\Delta s_{\lambda_{2,Y},i}\right\}. \quad (48)$$

In the previous equations, $s_{\mu_{Y},i}^{I}$ is the *i*-th interval sensitivity of the mean-value $\mu_{Y}(\boldsymbol{\alpha})$ of the response process $Y(\boldsymbol{\alpha},t)$ defined analytically by:

$$s_{\mu_{Y},i}^{I}=\frac{\partial\,\mu_{Y}(\boldsymbol{\alpha}^{I})}{\partial\Delta\alpha_{i}}\bigg|_{\Delta\boldsymbol{\alpha}=\mathbf{0}}=\Delta s_{\mu_{Y},i}\hat{e}_{i}^{I};\quad \Delta s_{\mu_{Y},i}=\mathbf{q}^{T}\,\mathbf{D}_{i}\,\boldsymbol{\mu}_{\mathbf{f}}; \quad (49)$$

and $s_{\lambda_{0,Y},i}^{I}$ and $s_{\lambda_{2,Y},i}^{I}$ are the interval sensitivities of the zero- and second-order spectral moments of the interval stochastic response process $Y(\boldsymbol{\alpha},t)$ given in explicit form by :

$$s_{\lambda_{\ell,Y},i}^{I}\equiv s_{\lambda_{\ell,\tilde{Y}},i}^{I}=\frac{\partial\lambda_{\ell,Y}(\boldsymbol{\alpha}^{I})}{\partial\Delta\alpha_{i}}\bigg|_{\Delta\boldsymbol{\alpha}=\mathbf{0}}=\Delta s_{\lambda_{\ell,Y},i}\,\hat{e}_{i}^{I},\quad \Delta s_{\lambda_{\ell,Y},i}=\mathbf{q}^{T}\,\mathbf{S}_{\boldsymbol{\Lambda}_{\ell,\mathbf{uu}},i}^{I}\,\mathbf{q};\quad \ell=0,2 \quad (50)$$

with $\mathbf{S}_{\boldsymbol{\Lambda}_{\ell,\mathbf{uu}},i}^{I}$ denoting the interval sensitivity of the spectral moments matrix of the random response, $\boldsymbol{\Lambda}_{\ell,\mathbf{uu}}(\boldsymbol{\alpha}^{I})$ (Muscolino et al., in press). Furthermore, in Eqs.(47) and (48), $\lambda_{\ell,Y}^{(0)}$, $\ell=0,2$, denote the spectral moments of the nominal system, which taking into account Eqs.(20) and (21), read:

$$\lambda_{\ell,Y}^{(0)}=\lambda_{\ell,Y}\left(\boldsymbol{\alpha}^{I}\right)\Big|_{\Delta\boldsymbol{\alpha}=\mathbf{0}}=2\int_{0}^{\infty}\omega^{\ell}\,\mathbf{q}^{T}\left[\mathbf{H}_{0}^{*}(\omega)\,\mathbf{G}_{\tilde{\mathbf{X}}_{\mathbf{f}}\tilde{\mathbf{X}}_{\mathbf{f}}}(\omega)\,\mathbf{H}_{0}^{T}(\omega)\right]\mathbf{q}\,\mathrm{d}\omega;\quad \ell=0,2. \quad (51)$$

Finally, the function $C\left(\beta_{0,Y}^{(0)},T\right)$ is defined as follows:

$$C\left(\beta_{0,Y}^{(0)},T\right)=-\frac{T}{4\,\pi}\frac{\lambda_{2,Y}^{(0)}}{\lambda_{0,Y}^{(0)}}\frac{L_{X_{\max}}^{(0)}\left(\beta_{0,Y}^{(0)},T\right)}{\sqrt{\lambda_{0,Y}^{(0)}\,\lambda_{2,Y}^{(0)}}}\exp\left(-\frac{\left(\beta_{0,Y}^{(0)}\right)^{2}}{2}\right). \tag{52}$$

In the case of randomly excited structural systems with interval uncertain parameters, the *CDF* turns out to be an interval function which can be written in the following form:

$$L_{\tilde{Y}_{\max}}^{I}(\tilde{b},T)=\left[\underline{L}_{\tilde{Y}_{\max}}(\tilde{b},T),\overline{L}_{\tilde{Y}_{\max}}(\tilde{b},T)\right] \tag{53}$$

where $\underline{L}_{\tilde{Y}_{\max}}(\tilde{b},T)$ and $\overline{L}_{\tilde{Y}_{\max}}(\tilde{b},T)$ are the *LB* and *UB* of $L_{\tilde{Y}_{\max}}^{I}(\tilde{b},T)$, respectively.

Based on Eq.(45) and applying the *improved interval analysis* (Muscolino and Sofi, 2011, 2012b), the *LB* and *UB* of the *CDF* of the *peak factor* can be evaluated as follows:

$$\begin{aligned}\underline{L}_{X_{\max}}\left(\boldsymbol{\alpha},\beta_{0,Y}^{(0)},T\right)&=L_{X_{\max}}^{(0)}\left(\beta_{0,Y}^{(0)},T\right)-\sum_{i=1}^{r}\left|\Delta s_{L_{X_{\max}},i}\left(\beta_{0,Y}^{(0)},T\right)\right|\Delta\alpha_i;\\ \overline{L}_{X_{\max}}\left(\boldsymbol{\alpha},\beta_{0,Y}^{(0)},T\right)&=L_{X_{\max}}^{(0)}\left(\beta_{0,Y}^{(0)},T\right)+\sum_{i=1}^{r}\left|\Delta s_{L_{X_{\max}},i}\left(\beta_{0,Y}^{(0)},T\right)\right|\Delta\alpha_i.\end{aligned} \tag{54a,b}$$

Once the interval *CDF* of the *peak factor* is known, the parameters of the response process needed to guarantee the desired safety level can be evaluated.

As a conclusion, it has to be emphasized that, under the assumption of slight deviation amplitudes of the uncertain parameters, i.e. $|\Delta\alpha_i| \ll 1$, since Eq.(45) gives the explicit expression of the interval reliability function, Eqs.(54a,b) provide the bounds of the interval reliability in explicit form too.

6. Numerical Application

The effectiveness of the proposed procedure is assessed by analyzing the 24-bar truss structure, depicted in Fig. 1, subjected to turbulent wind loads in the x -direction.

The Young's moduli of the diagonal bars are modeled as interval parameters $E_i^I=E_0\left(1+\Delta\alpha_i\hat{e}_i^I\right)$, $(i=16,17,...,24)$ with midpoint value $E_0=2.1\times10^{8}\,\mathrm{kN/m^2}$ and deviation amplitudes $\Delta\alpha_i=\Delta\alpha$ satisfying the condition $|\Delta\alpha|<1$. All the bars of the nominal structure are assumed to have cross-sectional area $A_{0,i}=A_0=5\times10^{-4}\,\mathrm{m^2}$ $(i=1,2,\ldots,24)$ while the nominal lengths of the bars $L_{0,i}(i=1,2,\ldots,24)$ can be deduced from Fig.1 where $L=3\,\mathrm{m}$. Furthermore, each node possesses a lumped mass $M=500\,\mathrm{kg}$. The Rayleigh damping constants c_0 and c_1 in Eq.(6) are taken as $c_0=3.517897\,\mathrm{s^{-1}}$ and $c_1=0.000547\,\mathrm{s}$, respectively, in such a way that the modal damping ratio for the first and third modes of the nominal structure is $\zeta_0=0.05$. The horizontal and vertical displacement random processes of the *j*-th node are denoted by $U_j(\boldsymbol{\alpha}^I,t)=\mu_{U_j}(\boldsymbol{\alpha}^I)+\tilde{U}_j(\boldsymbol{\alpha}^I,t)$ and $V_j(\boldsymbol{\alpha}^I,t)=\mu_{V_j}(\boldsymbol{\alpha}^I)+\tilde{V}_j(\boldsymbol{\alpha}^I,t)$, respectively.

The nodes $1,4$ and 7, located at different heights z_i $\left(i=1,4,7\right)$, are subjected to the nodal forces $F_{x,i}(z_i,t)$ in the along-wind direction (see Fig.1) expressed in the well-known form (Simiu and Scanlan, 1996):

$$F_{x,i}(z_i,t)=\frac{1}{2}\rho C_D A_i W^2(z_i,t)=F_{x,i}^{(s)}+\tilde{F}_{x,i}(z_i,t)\approx\frac{1}{2}\rho C_D A_i w_s^2+\rho C_D A_i \tilde{W}(z_i,t)w_s, \quad (i=1,4,7) \tag{55}$$

where $F_{x,i}^{(s)}$ and $\tilde{F}_{x,i}(z_i,t)$ denote the mean and random component of wind loads, related, respectively, to the mean-value, $w_s(z)$, and the fluctuating component, $\tilde{W}(z,t)$, of the wind velocity field $W(z,t)=w_s(z)+\tilde{W}(z,t)$.

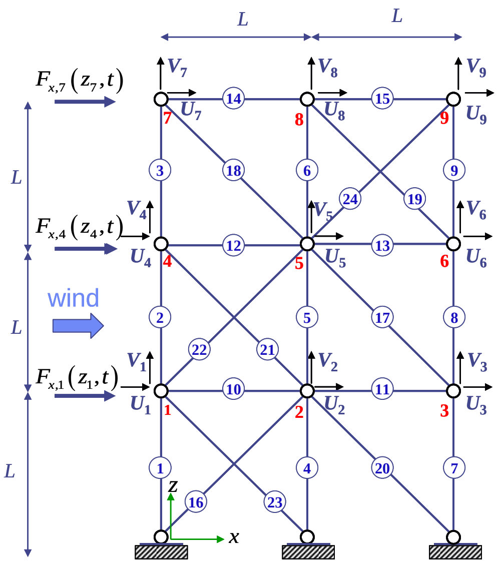
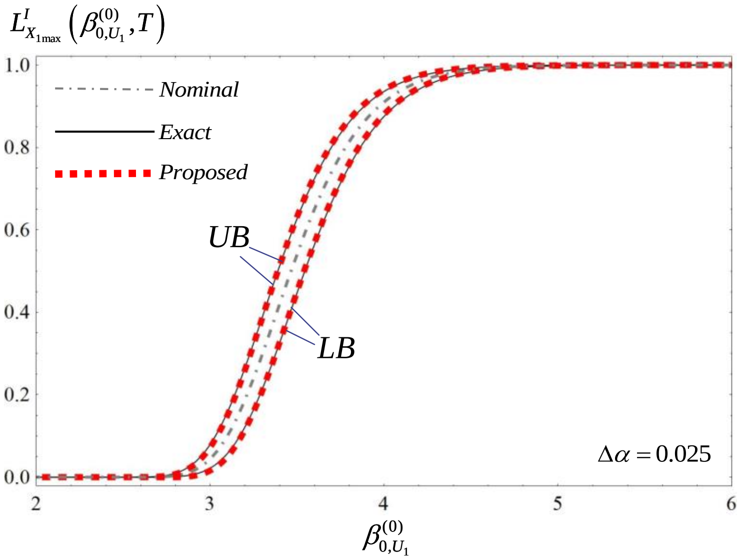


Figure 1. Truss structure under wind excitation.

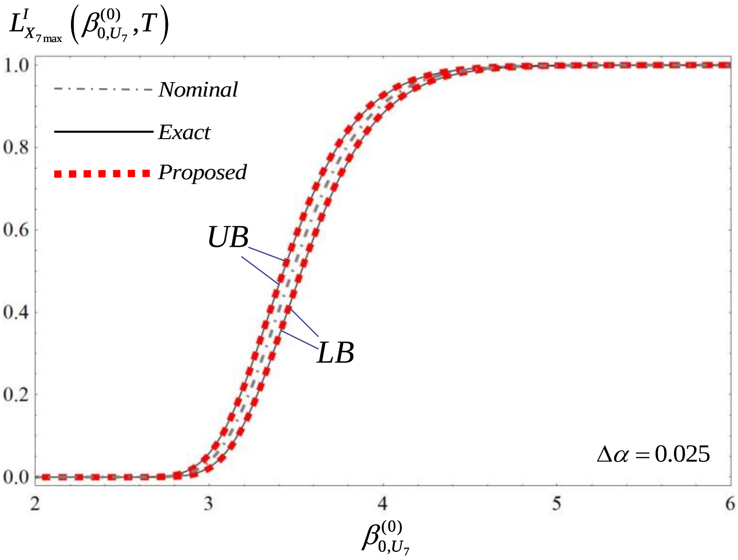
The mean wind velocity is assumed to vary with the elevation z following a power law, i.e. $w_s(z)=w_{s,10}\left(z/10\right)^{\gamma}$, where $w_{s,10}$ is the mean wind speed measured at height $z=10\,\text{m}$ above ground and γ is a coefficient depending on surface roughness, herein taken equal to $w_{s,10}=25\ \text{m/s}$ and $\gamma=0.3$, respectively. The fluctuating component $\tilde{W}(z,t)$ due to the turbulence in the flowing wind is modelled as a zero-mean stationary Gaussian random field, fully described from a probabilistic point of view by the one-sided *PSD* function $G_{\tilde{W}\tilde{W}}(\omega)$, defined here by the Davenport power spectrum (1961):

$$G_{\tilde{W}\tilde{W}}(\omega)=4K_0 w_{s,10}^2\frac{\chi^2}{\omega\left(1+\chi^2\right)^{4/3}} \tag{56}$$

where K_0 is the non-dimensional roughness coefficient, herein set equal to $K_0 = 0.03$, and $\chi = b_1\omega/(\pi w_{s,10})$ with $b_1 = 600\text{ m}$. Moreover, in Eq. (55) ρ is the air density; C_D is the drag coefficient; and A_i is the tributary area of the i-th node. These parameters are herein selected as follows: $\rho = 1.25\text{ Kg/m}^3$, $C_D = 1.2$, $A_1 = 9\text{ m}^2$, $A_4 = 9\text{ m}^2$ and $A_7 = 4.5\text{ m}^2$.

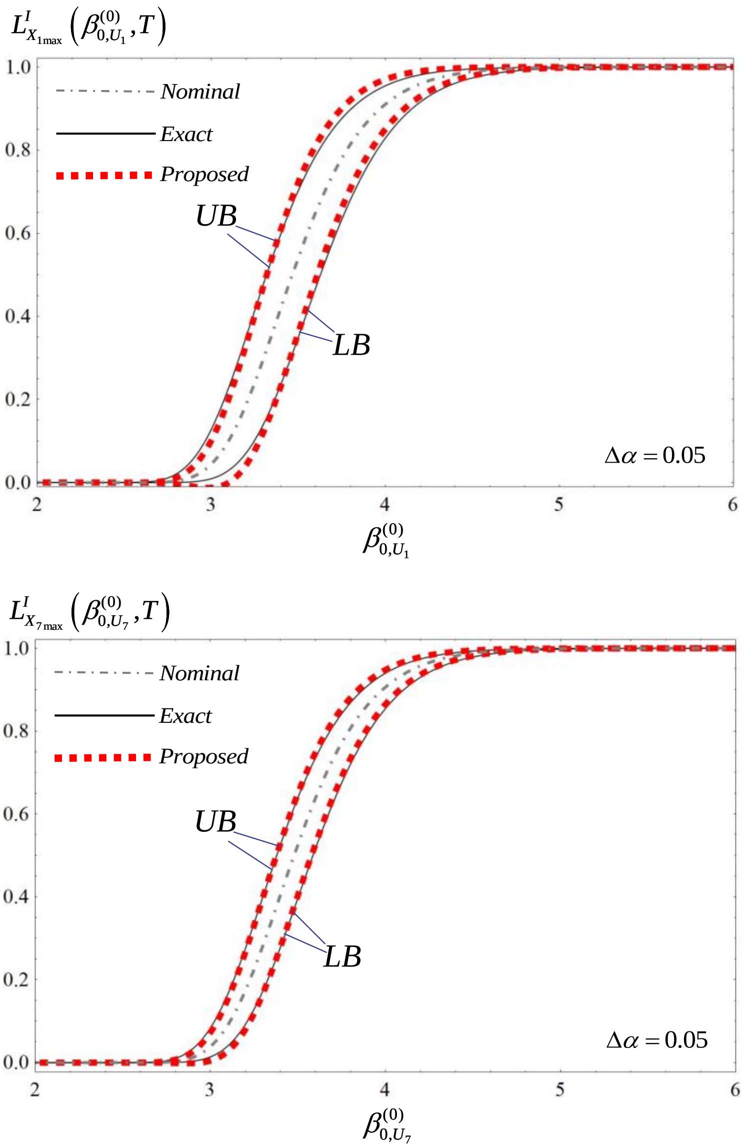


a)



b)

Figure 2. Comparison between the exact and proposed *UB* and *LB* of the *CDF* of the *peak factor processes* a) $X_{\max} = X_{1\max}$ and b) $X_{\max} = X_{7\max}$ of the horizontal displacements of nodes 1 and 7 of the truss structure with interval Young's moduli of the diagonal bars ($\Delta\alpha = 0.025$, $T = 1000T_0$).



a)

b)

Figure 3. Comparison between the exact and proposed *UB* and *LB* of the *CDF* of the *peak factor processes* a) $X_{\max} = X_{1\max}$ and b) $X_{\max} = X_{7\max}$ of the horizontal displacements of nodes 1 and 7 of the truss structure with interval Young's moduli of the diagonal bars ($\Delta\alpha = 0.05$, $T = 1000T_0$).

As outlined in the previous section, the *LB* and *UB* of the *CDF* can be evaluated by applying the *first-order interval Taylor series expansion* (see Eqs. (54a,b)). Figure 2 and 3 show the *LB* and *UB* of the interval *CDF* of the *peak factor processes* $X_{1\max}(\boldsymbol{\alpha}^I,T)$ and $X_{7\max}(\boldsymbol{\alpha}^I,T)$ of the horizontal displacements of nodes 1 and 7 of the truss structure (see Fig.1) for two different levels of uncertainty, namely $\Delta\alpha = 0.025$ and $\Delta\alpha = 0.05$, respectively. The estimates of the *LB* and *UB* provided by the *first-order* *interval Taylor series expansion* combined with the *improved interval analysis* are contrasted with the exact bounds. The latter are obtained following the philosophy of the so-called *vertex method* (Muhanna and Mullen, 2001; Moens and Vandepitte, 2005) consisting in evaluating the *CDF* of the selected *peak factor process* for all the combinations of the bounds of the uncertain parameters and then taking at each abscissa $\beta_{0,Y}^{(0)}$, with $Y = U_1, U_7$, the maximum and minimum value among all the *peak factor CDFs* so obtained. For completeness, the midpoint *CDF*, $L_{X_{\max}}^{(0)}\left(\beta_{0,Y}^{(0)}, T\right)$, with $X_{\max} = X_{1\max}, X_{7\max}$, of the selected displacement *peak factor processes* is also plotted. It is worth noting that for $\Delta\alpha = 0.025$ (see Fig. 2), the *UB* and *LB* of the *CDFs* provided by the proposed approach are almost coincident with the exact ones. Furthermore, as shown in Fig. 3, very accurate estimates of the bounds of the *CDFs* are also obtained for larger deviation amplitudes of the interval Young's moduli, say $\Delta\alpha = 0.05$.

7. Conclusions

An analytical approach to evaluate the reliability function for structures with uncertain-but-bounded parameters subjected to stationary Gaussian random excitation has been proposed.

The procedure has been developed within the framework of the widely used failure model identifying the probability of failure with the *first passage probability*, under the Poisson assumption of independent up-crossings. The interval reliability function has been evaluated in approximate closed-form by applying the *Interval Rational Series Expansion* in conjunction with the *improved interval analysis*, recently developed by the authors. The *Interval Rational Series Expansion* provides an approximate explicit expression of the inverse of an interval matrix with modifications. The *improved interval analysis* allows to limit the overestimation of the interval solution width due to the *dependency phenomenon* occurring in classical interval analysis.

Remarkable features of the proposed approach are: *i*) the capability of handling a large number of uncertainties and evaluate analytically the interval reliability function; *ii*) the possibility of providing very accurate explicit estimates of the bounds of the interval reliability in the framework of the *first-order interval Taylor series expansion*.

The presented procedure has been validated by analyzing a wind-excited truss structure with interval axial stiffness of the bars. Appropriate comparisons with the exact bounds obtained by the *vertex method* have shown that for small deviation amplitudes of the uncertain parameters, the *first-order interval Taylor series expansion* provides very accurate estimates of the region of the interval reliability.

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