Non-Linear Analysis of Beams with Large Deflections – An Interval Finite Element Approach

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Abstract: This work illustrates the development of Nonlinear Interval Finite Element Method for Euler-Bernouli Beams with large deflections under interval load. By including the von Karman strains (Reddy 2010), the secant stiffness is a function of the load. An interval load results in interval secant stiffness. Iterations using successively updated secant stiffness are used to obtain the large deflection solution. In this paper, the formulation of non-linear interval secant solution strategy is presented. Several example problems will be solved using the developed method. The behaviour of the solution is studied in terms of convergence, computational efficiency, and sharpness of interval bounds.

Keywords: Interval; Interval Finite Elements; Uncertainty; Nonlinear Strain

1. Introduction

Structures must be designed to safely respond to extreme loadings. Extreme loadings often results in larger deformations in structures which conflict with small strain analysis. Both analytical and numerical methods have been developed to perform structural analysis using large strains (Bathe 1996, Reddy 2010). To the authors knowledge, numerical methods have not been presented that will predict response of a structure, including nonlinear strain, when subject to uncertain interval loading. Of course, one could always conduct a Monte Carlo simulation to investigate the large strain response of a structure. However, MC simulations cannot provide guaranteed bounds on a solution while interval based methods can. In other applications, interval methods have been found to be computationally more efficient compared to conventional sensitivity methods or MC simulations. This paper is complementary to the authors' previous work on treating non-linear material behavior using interval description of uncertainty. In the modeling of non-linear materials, the constitutive relationships had behavior of a decreasing tangent, or secant stiffness as the material nonlinearities progressed. In the present model of a large deflection beam, the structure becomes stiffer as the loading increased. Thus, this paper expands nonlinear interval solutions methods to both softening and hardening systems.

In this paper, we will first review general interval methods for linear finite element methods. We will then formulate a large deflection beam in an interval sense. Sample calculations illustrating the application of the interval method are then presented along with observations on the behavior of the developed formulation. Finally, conclusions are given based on this work. .

2. A Review of Interval finite element method

Since the early development of Interval Finite Element Methods (IFEM) during the mid nineties of last century, several researchers worked on various aspects of Interval Finite Element Methods. (Alefeld and Herzberger, 1983; Neumaier, 1987; Köylüoglu, Cakmak, Ahmet, Soren, 1995; Koyluoglu, Cakmak and Nielsen,1995; Rao and Sawyer, 1995; Muhanna and Mullen,1995; Nakagiri and Yoshikawa,1996; Mullen and Muhanna,1996, Rao and Berke,1997;Koyluoglu, and Elishakoff, 1998; Rao and Chen, 1998,Nakagiri and Suzuki,1999; Muhanna and Mullen,1999; Mullen and Muhanna,1999). In particular, researchers have focused among other issues on two major problems; the first is how to obtain solutions for the resulting linear interval system of equations with reasonable bounds on the system response that make sense from practical point of view, or in other words with the least possible overestimation of their bounding intervals, the second is how to obtain reasonable bounds on the derived quantities that are functions of the system response. For example, when the system response is the displacement, the derived quantities might be forces or stresses which are functions of the displacements. Obtaining tight bounds on the derived quantities has been a tougher challenge due to the existing dependency of these quantities on the primary dependent variables which are already overestimated. So far, the derived quantities are obtained with significantly increased overestimation.

During the last decade, several significant advances have been made in the application of Interval Finite Element Methods (IFEM) to problems of uncertainty structural mechanics. Important contributions have been made by several researchers during the past one decade (Muhanna and Mullen, 2001;Corliss, Foley and Kearfott ,2004; Zhang,2005; Muhanna, Zhang and Mullen,2005; Popova, Iankov, Bonev, 2006;Neumaier and Pownuk ,2007; Rama Rao, Mullen and Muhanna,2010, Rama Rao, Mullen and Muhanna,2011). These researchers focused on the issue of computing structural response in the presence of uncertainty load, stiffness, and element cross sectional area.

A significant effort has been made in the work of Zhang (2005) to control the additional overestimation in the values of the derived quantities; the derived quantities have been calculated by an implicit substitution of the primary quantities. In addition to calculating rigorous bounds on the solution of the resulting linear interval system, a special treatment has been developed to handle the overestimation in the derived quantities. Instead of first evaluating the primary quantities and then substituting the obtained values in the expression for the derived quantities, the expression for the primary quantities has been substituted before its evaluation in the derived quantities expression and both were evaluated simultaneously preventing a large amount of overestimation in the values of derived quantities. In spite of the advancement provided by this approach, still it is conditioned by the original IFEM formulation and the special treatment of required transformations.

A significant improvement in the formulation of IFEM with application to truss problems has been introduced in the work of Neumaier and Pownuk (2007). This work has presented an iterative method for computing rigorous bounds on the solution of linear interval systems, with a computable overestimation factor that is frequently quite small. This approach has been demonstrated by solving truss problems with over 5000 variables and over 10000 interval parameters, with excellent bounds for up to about 10% input uncertainty. Although, no calculated derived quantities have been reported in this work, a formulation has been introduced for the calculation of derived quantities by intersecting the simple enclosure **z** = Z(**u**), where **z** depends linearly or nonlinearly on the solution **u** of the uncertain system with another enclosure obtained from the centered form (Neumaier and Pownuk, 2007, Eq. 4.13, pp 157). In spite of the provided improvement in this formulation, the two-step approach will result in additional overestimation when evaluating the derived quantities.

It is quite clear that among other factors, the issue of obtaining tight enclosures for the primary variables as well as for the derived quantities is conditioned by IFEM formulation and the methods used for the evaluation of the derived quantities. A new mixed formulation for Interval Finite Element Methods was developed by the authors (Rama Rao, Muhanna and Mullen, 2010, 2011) where the derived quantities of the conventional formulation are treated as dependent variables along with the primary variables. The formulation uses the mixed variational approach based on the Lagrange multiplier method. The system solution provides the primary variables along with the Lagrange multipliers which represent the derived quantities themselves.

3. Formulation

The kinematic equation of Euler-Bernoulli beam theory under the assumption that the beam is loaded in *x*-*y* plane of symmetry, the axial strain $\varepsilon_x(x,y)$ for large deformation (Bauchau and Craig, 2009) is expressed as follows:

$$\varepsilon_x(x,y) = \frac{\partial U}{\partial x} - y\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\left(\left(\frac{\partial U}{\partial x} - y\frac{\partial^2 V}{\partial x^2}\right)^2 + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2\right) \tag{1}$$

where *U* and *V* are the axial and transverse displacements, respectively.

Assuming $\left(\frac{\partial U}{\partial x} - y\frac{\partial^2 V}{\partial x^2}\right)^2$ to be small, the expression for $\varepsilon_x(x,y)$ is simplified as

$$\left.\begin{aligned} \varepsilon_x(x,y) &= \varepsilon_1(x,y) + \varepsilon_2(x,y) \\ \varepsilon_1(x,y) &= \frac{\partial U}{\partial x} - y\frac{\partial^2 V}{\partial x^2} \\ \varepsilon_2(x,y) &= \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2 \end{aligned}\right\} \tag{2}$$

Within the context of finite element, the shape functions of the axial displacement can be approximated by a linear function and those of the transversal displacement by a cubic one. Thus the shape functions for cubic-linear beam are defined as

$$\left.\begin{aligned} U(x) &= \frac{(L-x)}{L}U_1 + \frac{x}{L}U_2 \\ V(x) &= [N_1]\{d\} \end{aligned}\right\} \tag{3}$$

where $$N_1 = \begin{bmatrix} 0 & \frac{(2x^3-3x^2+L^3)}{L^3} & \frac{(x^3L-2x^2L^2+xL^3)}{L^3} & 0 & \frac{(-2x^3+3x^2L)}{L^3} & \frac{(x^3L-x^2L^2)}{L^3} \end{bmatrix} \tag{4}$$

and $$\{d\} = [U_1 \quad V_1 \quad \Phi_1 \quad U_2 \quad V_2 \quad \Phi_2]^T \tag{5}$$

The first derivative of U and first and second derivatives of V are given as

$$\left.\begin{aligned} \frac{\partial U}{\partial x} &= \begin{bmatrix} -\frac{1}{L} & 0 & 0 & \frac{1}{L} & 0 & 0 \end{bmatrix}\{d\} \\ \frac{\partial V}{\partial x} &= \left[N_1^{'}\right]\{d\} \\ \frac{\partial^2 V}{\partial x^2} &= \left[N_1^{''}\right]\{d\} \end{aligned}\right\} \tag{6}$$

where $$\left[N_1^{'}\right] = \begin{bmatrix} 0 & \frac{(6x^2-6xL)}{L^3} & \frac{(3x^2L-4xL^2+L^3)}{L^3} & 0 & \frac{(-6x^2+6xL)}{L^3} & \frac{(3x^2L-2xL^2)}{L^3} \end{bmatrix} \tag{7}$$

and $$\left[N_1^{''}\right] = \begin{bmatrix} 0 & \frac{(12x-6L)}{L^3} & \frac{(6xL-4L^2)}{L^3} & 0 & \frac{(-12x+6L)}{L^3} & \frac{(6xL-2L^2)}{L^3} \end{bmatrix} \tag{8}$$

Using Eqs. 3, 7, and 8 $\varepsilon_1(x,y)$ and $\varepsilon_2(x,y)$ can be expressed as

$$\left.\begin{aligned} \varepsilon_1(x,y) &= [B_1]\{d\} \\ \varepsilon_2(x,y) &= \frac{1}{2}\{d\}^T[B_2]\{d\} \end{aligned}\right\} \tag{9}$$

where

$$\left.\begin{aligned} [B_1] &= \begin{bmatrix} -\frac{1}{L} & -\frac{(12x-6L)}{L^3}y & -\frac{(6xL-4L^2)}{L^3}y & \frac{1}{L} & -\frac{(-12x+6L)}{L^3}y & -\frac{(6xL-2L^2)}{L^3}y \end{bmatrix} \\ [B_2] &= \left[N^{'}\right]^T\left[N^{'}\right] \end{aligned}\right\} \tag{10}$$

The axial strain $\varepsilon_1(x,0)$ at the neutral axis is obtained by substituting y = 0 in Eq. (10) as

$$\varepsilon_1(x,0) = [B_{1mid}]\{d\} \tag{11}$$

where $[B_{1mid}] = \begin{bmatrix} -\frac{1}{L} & 0 & 0 & \frac{1}{L} & 0 & 0 \end{bmatrix}$ (12)

The total potential energy Π of a beam with geometric strain effects and subjected to surface traction q on portion L_1 of the surface is given by

$$\Pi = \frac{1}{2}\int_{x=0}^{L}\int_A \varepsilon_x(x,y)E\varepsilon_x(x,y)dAdx - \int_{L_1} qdx \tag{13}$$

Eq. (2) can be used in Eq. (13) to obtain

$$\Pi = \frac{1}{2}\int_{x=0}^{L}\int_A \left(\varepsilon_1(x,y)+\varepsilon_2(x,y)\right)E\left(\varepsilon_1(x,y)+\varepsilon_2(x,y)\right)dAdx - \int_{L_1} qdx \tag{14}$$

This can be expanded as

$$\left.\begin{aligned} \Pi = \frac{1}{2}\int_{x=0}^{L}\int_A \varepsilon_1(x,y)E(x,y)\varepsilon_1(x,y)dAdx + \int_{x=0}^{L}\int_A \varepsilon_1(x,y)E(x,y)\varepsilon_2(x,y)dAdx + \\ \frac{1}{2}\int_{x=0}^{L}\int_A \varepsilon_2(x,y)E(x,y)\varepsilon_2(x,y)dAdx - \int_{L_1} qdx \end{aligned}\right\} \tag{15}$$

or

$$\Pi = \Pi_1 + \Pi_2 + \Pi_3 - \int_{L_1} qdx$$

The discrete structural model can be given as

$$\Pi = \frac{1}{2}\{d^T\}K\{d\} - \{d^T\}P$$

In order to obtain the stiffness matrix K for the nonlinear case, from equation (9) we obtain:

$$\Pi_1 = \frac{1}{2}\int_{x=0}^{L}\int_A \{d\}^T\left[B_1^T\right]E(x,y)[B_1]\{d\}dAdx$$

$$\Pi_2 = \frac{1}{2}\int_{x=0}^{L}\int_A \{d\}^T\left[B_1^T\right]E(x,y)\{d^T\}[B_2]\{d\}dAdx$$

$$\Pi_3 = \frac{1}{8}\int_{x=0}^{L}\int_A \{d\}^T\left[B_2^T\right]\{d\}E(x,y)\{d\}^T\left[B_2\right]\{d\}dAdx$$

Invoking $\delta\Pi = 0$ and dropping the load term for simplicity, this gives

$$\delta\Pi = \{\delta d_i\}^T \int_{x=0}^{L}\int_A B_1^T E(x,y) B_1 dAdx \{d_j\} +$$
$$\frac{3}{4}\{\delta d_i\}^T \int_{x=0}^{L}\int_A B_1^T E(x,y) B_2 dAdx \{d_j\} + \frac{3}{4}\{\delta d_i\}^T \int_{x=0}^{L}\int_A B_2^T E(x,y) B_1 dAdx \{d_j\} +$$
$$\frac{1}{2}\{\delta d_i\}^T \int_{x=0}^{L}\int_A B_2^T E(x,y) B_2 dAdx \{d_j\} = 0$$

Since δd is arbitrary, we obtain

$[K]\{d\} = P$

Where $K = K_1+K_2+K_3$ and

$$[K_1] = \int_{x=0}^{L}\int_A B_1^T E(x,y) B_1 dAdx$$

$$[K_2] = \frac{3}{4} AE \int_{x=0}^{L} \left[B_1^T B_2 + B_2^T B_1\right] dx$$

$$[K_3] = \frac{1}{2} AE \int_{x=0}^{L} B_2^T B_2 dx$$

In the current interval formulation, the load is assumed to be given in an interval form which results in an interval solution for displacements $\{\mathbf{d}\}$ and consequently interval strains $\boldsymbol{\varepsilon}$. Here stiffness matrix is obtained using numerical integration, in order to avoid membrane locking, K_2 and K_3 are evaluated using reduced integration in x direction only. Now we rewrite $\boldsymbol{\varepsilon}_1(x,y)$ and $\boldsymbol{\varepsilon}_2(x,y)$ in Eq. (9) as interval

$$\left.\begin{array}{l} \boldsymbol{\varepsilon}_1(x,y) = [B_1]\{\boldsymbol{d}\} \\ \boldsymbol{\varepsilon}_1(x) = [B_{1mid}]\{\boldsymbol{d}\} \\ \boldsymbol{\varepsilon}_2(x) = \frac{1}{2}\{\boldsymbol{d}\}^T [B_2]\{\boldsymbol{d}\} = \frac{1}{2}\{\boldsymbol{d}\}^T \left[N'\right]^T \left[N'\right]\{\boldsymbol{d}\} = \boldsymbol{\eta}\left[N'\right]\{\boldsymbol{d}\} \\ \text{where } \boldsymbol{\eta} \text{ is a scalar interval multiplier} \end{array}\right\} \quad (16)$$

In light of the reduced integration of second and third terms of Eq. (16), the individual stiffness matrices can be rewritten as

$$\left.\begin{array}{l} [K_1] = \int_{x=0}^{L}\int_A [B_1]^T E [B_1] dAdx \\ [K_2] = \frac{3}{4} A\left(\int_{x=0}^{L} [B_{1mid}]^T \boldsymbol{\eta} E \left[N'\right] dx + \int_{x=0}^{L} \left[N'\right]^T \boldsymbol{\eta} E [B_{1mid}] dx\right) \\ [K_3] = \frac{1}{2} A \int_{x=0}^{L} \left[N'\right]^T \boldsymbol{\eta}^2 E \left[N'\right] dx \end{array}\right\} \quad (17)$$

Thus K_1 matrix in the present work is computed using 2×2 integration points, K_2 and K_3 matrices are evaluated using reduced integration at one point. Matrices K_1 , K_2 and K_3 can be shown in a simplified form as follows:

$$\left.\begin{aligned} [K_1] &= \left[A_1^T\right][D_1][A_1] \\ [K_2] &= \frac{3}{4}\left[A_1^T\right][\boldsymbol{\eta}][D_{1mid}][A_2] + 1.5\left[A_2^T\right][\boldsymbol{\eta}][D_{1mid}][A_1] \\ [K_3] &= \frac{1}{2}\left[A_2^T\right][\boldsymbol{\eta^2}][D_{1mid}][A_2] \end{aligned}\right\} \tag{18}$$

Using Eq. (18), the overall stiffness matrix K can be defined as follows

$$[K] = \begin{bmatrix} A_1^{\boldsymbol{T}} & A_1^{\boldsymbol{T}} & A_2^{\boldsymbol{T}} & A_2^{\boldsymbol{T}} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \boldsymbol{\eta} & & \\ & & \boldsymbol{\eta} & \\ & & & \boldsymbol{\eta}^2 \end{bmatrix} \begin{bmatrix} D_1 & & & \\ & D_{1\boldsymbol{mid}} & & \\ & & D_{1\boldsymbol{mid}} & \\ & & & D_{1\boldsymbol{mid}} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_1 \\ A_2 \end{bmatrix} \tag{19}$$

where $[D_1]$ is a diagonal matrix whose diagonal terms are the product of weights w and E at each of m×n integration points used for the double integral. Also $[D_{1mid}]$ is a diagonal matrix whose diagonal terms are the product of weights w and E at each of the p integration points used for the single integral. Each entry $\boldsymbol{\eta}$ occurring in the interval diagonal matrix contains the interval multipliers at each of the p integration points used for the single integral.

This can be shown as

$$[K] = [G_1][\boldsymbol{\eta}][D][G_2] \tag{20}$$

Considering the mixed finite element formulation developed earlier by the authors (Rama Rao, Muhanna and Mullen, 2011), the following system of equations is considered:

$$\begin{bmatrix} K & C^T \\ C & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{P} \\ \boldsymbol{0} \end{Bmatrix} \tag{21}$$

Here C is the constraint matrix, $\boldsymbol{\lambda}$ is the vector of interval multipliers, **P** is the vector of external loads and **U** is the vector of interval nodal displacements. Substituting Eq. (20) in Eq. (21), we obtain

$$\begin{bmatrix} [G_1][\boldsymbol{\eta}][D][G_2] & C^T \\ C & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{P} \\ \boldsymbol{0} \end{Bmatrix} \tag{22}$$

It is observed that Eq. (22) has the structure $B^T DA$ given by Neumaier (Neumaier and Pownuk, 2007).

4. Example problems

Two example problems are chosen to illustrate the applicability of the present interval approach to handle geometric nonlinearity in case of beam element. These examples are chosen to demonstrate the ability of the current approach to obtain sharp bounds to the displacements and forces even in the presence of large uncertainties and large number of interval variables. It is assumed that for each element in the structure obeys a linear constitutive relationship. Interval uncertainty of load is considered for the two example problems. The example problems are solved for various levels of interval widths of the loads centred at their nominal values. All interval variables are assumed to vary independently. Two approaches are used, namely; interval and combinatorial.

The first example chosen is a fixed-fixed beam of L=100 in. (2.54m) span as shown in Figure 1. Four elements consecutively numbered from left to right are used. The beam is subjected to a nominal distributed load of *w*=10 lb/in. (1.751 kN/m) in the vertically downward direction along the entire span. The material properties for each element are given in Table 1, while the cross sectional dimensions are 1in.×1in. (0.0254m×0.0254m). All elements have a modulus of elasticity of 30 ksi (206.84 GPa). Solution is obtained by following the procedure outlines in the earlier section. For double integral, two integration points are used along the length of the each element and two integration points across the height. For reduced integration using single integral, one integration point is chosen at the level of the neutral axis of the beam along the direction of span. A linear constitutive model is used to analyze this example.

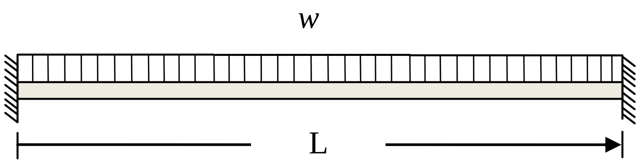

Figure 1. Fixed-Fixed beam subjected to distributed loading

In order to validate the present formulation, the beam is subjected to an incremental loading from 0 to 10 lb/in. in steps 1 lb/in. Considering zero load uncertainty, the vertical displacements of the mid-span of the beam are compared to those given by Reddy (2010) and are plotted in Figure 2 . It is clearly observed from the Figure 2 that the vertical displacements computed by the present approach closely match the results obtained by Reddy.

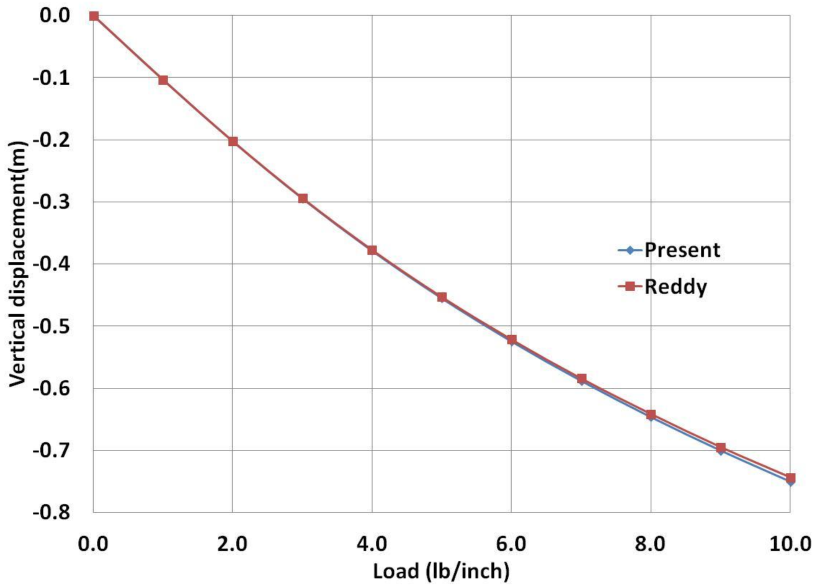

Figure 2 Fixed-Fixed beam- vertical displacements at mid-span at various levels of loading

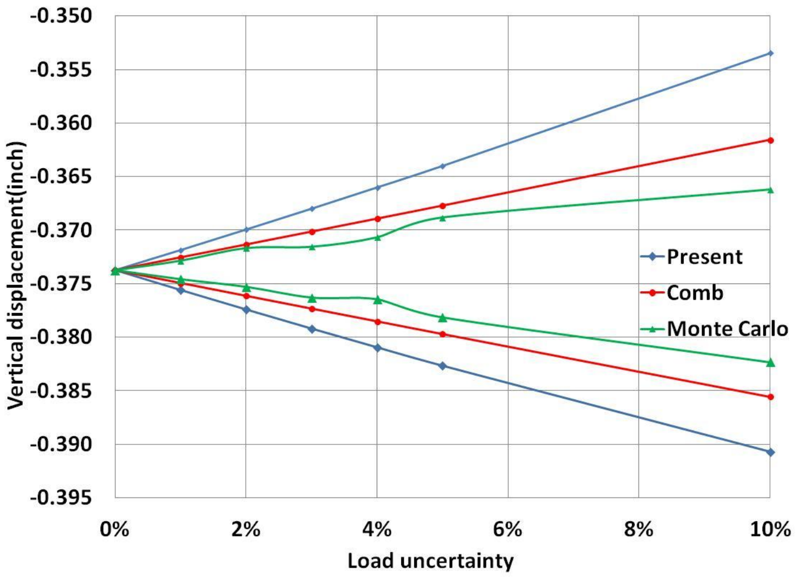

Figure 3. Fixed-Fixed beam- vertical displacements at mid-span at various levels of uncertainty with 10 lb/in. loading

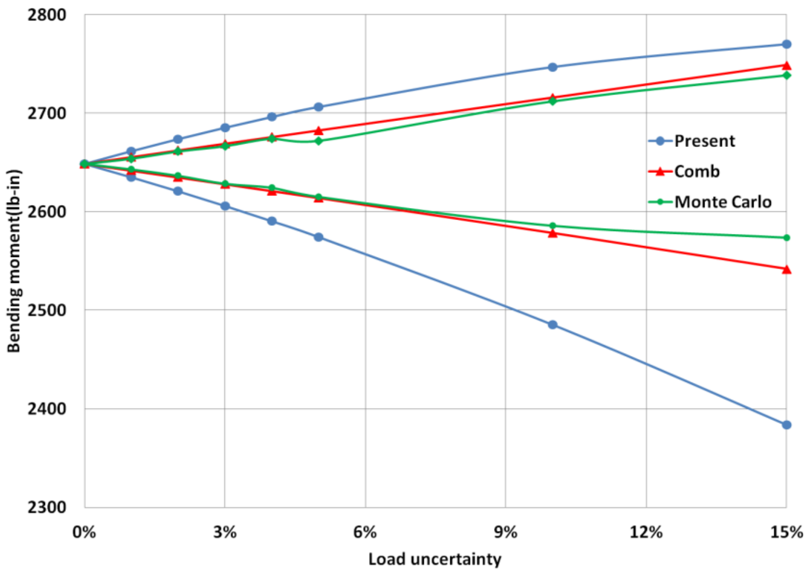

Figure 4. Fixed-Fixed beam- bending moment at mid-span at various levels of uncertainty with 10 lb/in. loading

Figure 3 shows the plot of vertical displacement of mid-span for a load of 10 lb/in. acting all over the span using the present approach, combinatorial approach and Monte Carlo simulation approach (100 realization) at various levels of uncertainty. It is observed from Figure 3, that the present solution encloses the combinatorial solution sharply and Monte Carlo simulation provides sharp inner bounds to the combinatorial solution at various levels of uncertainty. Figure 4 shows a plot of the mid-span bending moment for the same case. It is also observed from Figure 4 that the combinatorial solution is enclosed sharply from inside and outside by the Monte Carlo solution and the present interval solution respectively.

The second example chosen is a beam with pinned-pinned supports at both ends as shown in the Figure 5. It has the same geometric and material properties and same loading as that of the fixed beam cited in the first example.

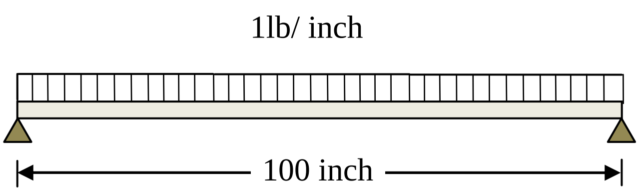

Figure 5. Pinned-Pinned beam subjected to distribute load

Figure 6 shows the plot of vertical displacement of mid-span for a load of 1 lb/in. acting all over the span using the present approach, combinatorial approach and Monte Carlo simulation approach at various levels of uncertainty. It is observed from Figure 6, that the present solution encloses the combinatorial solution

sharply and Monte Carlo simulation provides sharp inner bounds to the combinatorial solution at various levels of uncertainty. Figure 7 shows a plot of the mid-span bending moment for the same case. It is also observed from Figure 7 that the combinatorial solution is enclosed sharply from inside and outside by the Monte Carlo solution and the present interval solution respectively.
However, it is observed that the present formulation is not able to able to provide an enclosure to combinatorial solution at higher loads in case of pinned beam. Efforts are on to investigate this issue.

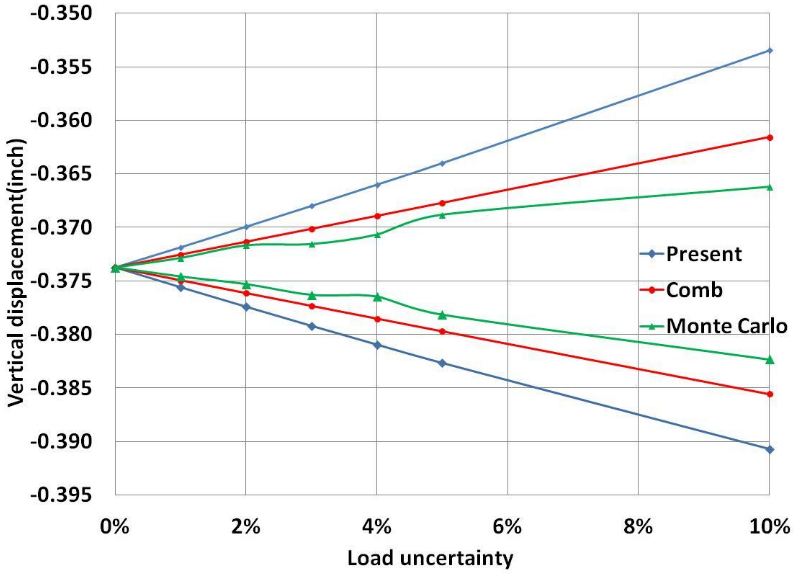

Figure 6. Pinned-Pinned Beam - Vertical displacement at mid-span at various levels of uncertainty with 1 lb/inch loading

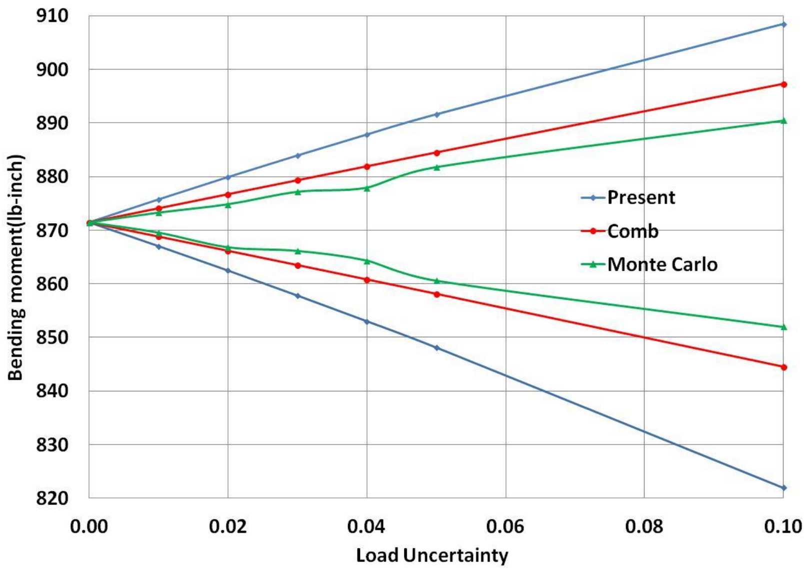

Figure 7. Pinned-Pinned beam- Bending moment at mid-span at various levels of uncertainty with 1 lb/in. loading

5. Conclusions

A new formulation for interval analysis of geometrically nonlinear beam is presented. The present approach is validated by comparing it with the results obtained earlier by Reddy. Solutions are computed using the present approach, combinatorial approach and Monte Carlo simulation. It is observed that the present solution encloses the combinatorial solution at various levels of uncertainty.

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