# Recent Advances in Reliability Estimation of Time-Dependent Problems Using the Concept of Composite Limit State

Z. P. Mourelatos, M. Majcher, V. Pandey
*Mechanical Engineering Department, Oakland University; mourelat, mtmajch2, pandey2@oakland.edu*

**Abstract:** A new reliability analysis method is proposed for time-dependent problems with limit-state functions of input random variables, input random processes and explicit in time using the total probability theorem and the concept of composite limit state. The input random processes are assumed Gaussian. They are expressed in terms of standard normal variables using a spectral decomposition method. The total probability theorem is employed to calculate the time-dependent probability of failure using a time-dependent conditional probability which is computed accurately and efficiently in the standard normal space using FORM and a composite limit state of linear instantaneous limit states. If the dimensionality of the total probability theorem integral (equal to the number of input random variables) is small, we can easily calculate it using Gauss quadrature numerical integration. Otherwise, simple Monte Carlo simulation or adaptive importance sampling is used based on a pre-built Kriging metamodel of the conditional probability. An example from the literature on the design of a hydrokinetic turbine blade under time-dependent river flow load demonstrates all developments.

**Keywords:** Time-dependent probability of failure; System Reliability; Random process quantification; Composite limit state; Convex polyhedral safe sets

## 1. Introduction

The response of time-dependent systems under uncertainty is a random process. The input commonly consists of a combination of random variables and random processes. A time-dependent reliability analysis is thus, needed to calculate the probability that the system will perform its intended function successfully for a specified time.

Reliability is an important engineering requirement for consistently delivering acceptable product performance through time. As time progresses, the product may fail due to time-dependent operating conditions and material properties, component degradation, etc. The reliability degradation with time may increase the lifecycle cost due to potential warranty costs, repairs and loss of market share. In this article, we use time-dependent reliability concepts associated with the first-passage of non-repairable systems. Among its many applications, the time-dependent reliability concept can be used to reduce the lifecycle cost (Singh, Mourelatos and Li, 2010a) or to set a schedule for preventive condition-based maintenance (Singh, Mourelatos and Li, 2010b).

The time-dependent probability of failure, or cumulative probability of failure (Singh, Mourelatos, Li, 2010a; Andrieu-Renaud, Sudret and Lemaine, 2004), is defined as

$$P_f(0,T) = P\{\exists t \in [0,T] : g(\mathbf{X}, \mathbf{Y}(t), t) \le 0\} \tag{1}$$

where the limit state $g(\mathbf{X}, \mathbf{Y}(t), t) = 0$ depends on the vector $\mathbf{X} = [X_1 \quad X_2 \quad \cdots \quad X_n]$ of $n$ input random variables, the vector $\mathbf{Y}(t) = [Y_1(t) \quad Y_2(t) \quad \cdots \quad Y_m(t)]$ of $m$ input random processes, and explicit time $t$. Failure occurs if $g(\cdot) \le 0$ at any time $t \in [0, T]$ where $T$ is the planning horizon (time of interest).

The random process $g(\mathbf{X}, \mathbf{Y}(t), t)$ can be viewed as a collection of random variables at different time instances $t$. Since we consider a first excursion failure problem in Eq. (1), the failure domain is defined as

$$F = \left\{ \max_{t \in [0,T]} g(\mathbf{X}, \mathbf{Y}(t), t) \le 0 \right\}.$$

(2)

The time-dependent probability of failure of Eq. (1) can be calculated exactly as

$$P_f(0,T) = 1 - \left(1 - P_f^i(0)\right) \exp\left\{ - \int_0^T \lambda(t)\,\mathrm{d}t \right\} \tag{3}$$

where $P_f^i(0)$ is the instantaneous probability of failure at the initial time and

$$\lambda(t) = \lim_{\mathrm{d}t \to 0} \frac{P(t < T_f \le t + \mathrm{d}t \mid T_f > t)}{\mathrm{d}t} \tag{4}$$

is the failure rate with $T_f$ denoting the time to failure.

In the commonly used out-crossing rate approach, the failure rate is approximated by the up-crossing rate

$$\nu^+(t) = \lim_{\Delta\tau \to 0} \frac{P\left[g(\mathbf{X}, \mathbf{Y}(t), t) > 0 \cap g(\mathbf{X}, \mathbf{Y}(t+\Delta\tau), t+\Delta\tau) \le 0\right]}{\Delta\tau} \tag{5}$$

under the assumptions that the probability of having two or more out-crossings in $[t, t+\Delta t]$ is negligible, and the out-crossings in $[t, t+\Delta t]$ are statistically independent of the previous out-crossings in $[0, t]$ and therefore, Poisson distributed.

Monte Carlo simulation (MCS) can accurately estimate the probability of failure of Eq. (1) but it is computationally prohibitive for dynamic systems with a low failure probability. To address the computational issue of MCS, analytical methods have been developed based on the out-crossing rate approach which was first introduced by Rice (Rice, 1954) followed by extensive studies (Andrieu-Renaud, Sudret and Lemaine, 2004; Rackwitz, 1998; Sudret, 2008; Zhang and Du, 2011). The PHI2 method (Andrieu-Renaud, Sudret and Lemaine, 2004) uses two successive time-invariant analyses based on FORM, and the binomial cumulative distribution to calculate the probability of the joint event in Eq. (5). A Monte-Carlo based set theory approach has been also proposed (Savage and Son, 2009; Son and Savage, 2007) using a similar approach with the PHI2 method. Because analytical studies such as in (Singh and Mourelatos, 2010; Hu and Du, 2013; Hu, Li, Du and Chandrashekhara, 2012) have shown that the PHI2-based approach lacks sufficient accuracy for vibratory systems, analytical approaches have been proposed to accurately estimate the time-dependent probability of failure considering non-monotonic behavior (Singh and Mourelatos, 2010; Li and Mourelatos, 2009; Hu and Du, 2012).

The limited accuracy of the out-crossing rate approach has been recently improved by considering the correlations between the limit-state function at two time instants (Hu, Li, Du and Chandrashekhara, 2012). The method estimates the up-crossing rate $\nu^{+}(t)$ by solving an integral equation involving $\nu^{+}(t)$ and $\nu^{++}(t, t_1)$, the joint probability of up-crossings in times $t$ and $t_1$ (Madsen and Krenk, 1984).

Among the simulation-based methods, a MCS approach was proposed in (Singh and Mourelatos, 2010) to estimate the time-dependent failure rate over the product lifecycle and its efficiency was improved using an importance sampling method with a decorrelation length (Singh, Mourelatos and Nikolaidis, 2011) in order to reduce the high dimensionality of the problem. Subset simulation (Au and Beck, 2001; Au and Beck, 2003) has been recently developed as an efficient simulation method for computing small failure probabilities for general reliability problems. Its efficiency comes from introducing appropriate intermediate failure sub-domains to express the low probability of failure as a product of larger conditional failure probabilities which are estimated with much less computational effort. Because it is very challenging to generate samples in the conditional spaces, subset simulation with Markov Chain Monte Carlo (SS/MCMC) (Beck and Au, 2002) and subset simulation with splitting (SS/S) (Ching, Beck and Au, 2005; Wang, Mourelatos, Li, Singh and Baseski, 2013) methods have been introduced. Extreme value methods have also been proposed (Wang and Wang, 2012; Chen and Li, 2007) using the distribution of the extreme value of the response. The distribution estimation however, is computationally expensive requiring for example, advanced metamodels considering the time dimension and expensive global optimization (Wang and Wang, 2012).

In this paper, we present a time-dependent reliability analysis using the total probability theorem. For each realization of the random variables in estimating the integral expressing the total probability theorem, we calculate a conditional time-dependent probability of failure using the concept of composite limit state and FORM. An advantage of our approach is that the conditional probability is very simple to calculate especially if the instantaneous limit states for the conditional probability are linear. Also, we can easily handle non-normal and correlated random variables without additional computational effort. The evaluation of the integral expressing the total probability theorem determines however, the accuracy and efficiency of the method. For that, we can use numerical integration for low dimensional problems and MCS or adaptive importance sampling for high dimensional problems based on a metamodel of the conditional probabilities. We present details and discuss future work.

The paper is arranged as follows. Section 2 describes the proposed methodology including the spectral decomposition of the input random processes and the composite limit state approach to calculate the time-dependent conditional probabilities for the total probability theorem. Section 3 uses the design of a hydrokinetic turbine blade to demonstrate all developments. Finally, Section 4 summarizes, concludes and highlights future research.

## 2. Proposed Methodology

### 2.1. SPECTRAL DECOMPOSITION OF RANDOM PROCESSES

We assume that each $Y(t)$ in Eq. (1) is a Gaussian, non-stationary random process defined by the mean function $\mu_Y(t)$, the standard deviation function $\sigma_Y(t)$ and the autocorrelation function $\rho_Y(t_1,t_2)$. The time interval $[0,T]$ is discretized using $N$ discrete times $t_1=0, t_2, \ldots, t_N=T$ and the $N \times N$ covariance matrix $\mathbf{\Sigma}=\lfloor Cov(t_i,t_j)\rfloor$, $i=1,2,\ldots,N$; $j=1,2,\ldots,N$ is formed where $Cov(t_i,t_j)=\sigma_Y(t_i)\sigma_Y(t_j)\rho_Y(t_i,t_j)$ is the covariance between times $t_i$ and $t_j$.

Let $\mathbf{\Sigma}=\mathbf{\Phi}\cdot\mathbf{\Lambda}\cdot\mathbf{\Phi}^T$ be the spectral decomposition of the covariance matrix $\mathbf{\Sigma}$ where $\mathbf{\Phi}=[\mathbf{\Phi}_1 \quad \mathbf{\Phi}_2 \quad \cdots \quad \mathbf{\Phi}_N]$ is the orthonormal matrix of the eigenvectors $\mathbf{\Phi}_i, i=1,\ldots,N$ and $\mathbf{\Lambda}=diag[\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_N]$ is the diagonal matrix of the corresponding eigenvalues. Also, let $\mathbf{Z}=(Z_1 \quad Z_2 \quad \cdots \quad Z_N)$ be a vector of $N$ independent standard normal variables. Because of the affine transformation property of the multi-normal distribution, the following spectral representation holds

$$Y(t)=\mu_Y(t)+\sum_{i=1}^{r}\sqrt{\lambda_i}\cdot\mathbf{\Phi}_i^T(t)\cdot Z_i \tag{6}$$

where $t=t_1, t_2, \ldots, t_N$ and $r \le N$ is the number of dominant eigenvalues. Eq. (6) is used to generate sample functions (trajectories) of $Y(t)$. The Expansion Optimal Linear Estimation method (EOLE) (Li and Kiureghian, 1993) can also be used. Eq. (6) indicates that each random variable $Y_i=Y(t_i)$, $i=1,\ldots,N$ is represented by a linear combination of standard normal random variables $Z_i, i=1,\ldots,r$.

### 2.2. TIME-DEPENDENT RELIABILITY AND TOTAL PROBABILITY THEOREM

Let $F=\{\mathbf{x}, \mathbf{y}(t): g(\mathbf{X},\mathbf{Y}(t),t)\le 0,\ t\in[0,\ T]\}$ be the failure event where $\mathbf{x}, \mathbf{y}(t)$ are realizations of $\mathbf{X}$ and $\mathbf{Y}(t)$, respectively. According to the total probability theorem, the time-dependent probability of failure of Eq. (1) can be expressed as

$$P_f(0,T)=P(F)=\int_{\Omega} P\left(F/\mathbf{W}\right) f_{\mathbf{W}}(\mathbf{w})d\mathbf{w} \tag{7}$$

where $P\left(F/\mathbf{W}\right)$ is a time-dependent conditional probability of failure, $f_{\mathbf{W}}(\mathbf{w})$ is the joint PDF of the input random variables $\mathbf{W}$ and $\Omega$ is the support of $f_{\mathbf{W}}(\mathbf{w})$. The set $\mathbf{W}$ is a proper subset of $[\mathbf{X} \quad \mathbf{Y}(t)]$ $(\mathbf{W}\subset[\mathbf{X} \quad \mathbf{Y}(t)])$ and is chosen such that the time-dependent $P\left(F/\mathbf{W}\right)$ can be calculated accurately and efficiently.

The integral of Eq. (7) is calculated using numerical integration schemes such as adaptive Simpson's rule or adaptive Gauss-Lobatto rule (Abramowitz and Stegun, 1965) if the number *n* of random variables is small (e.g. less than 5). Otherwise, it is estimated using simple Monte Carlo simulation or regular (or adaptive) importance sampling methods. In all cases, $P\left(F/_{\mathbf{W}}\right)$ is calculated directly or based on a pre-built metamodel.

Our approach is based on the efficient and accurate calculation of $P\left(F/_{\mathbf{W}}\right)$ as described in Section 2.3. An advantage of the total probability theorem of Eq. (7) is that non-normal and correlated random variables are handled without additional computational effort or loss of accuracy.

### 2.3. Calculation of Time-Dependent Conditional Probability of Failure

This section describes how to calculate the time-dependent probability of failure $P\left(F/_{\mathbf{W}}\right)$ in Eq. (7). We first choose the set **W** so that for each realization **w** of **W**, the limit state becomes preferably (but not necessarily) a linear combination of normal random variables at each time *t*. For example, let $g(\mathbf{X},\mathbf{Y}(t),t)=$ $=X_1^3+2X_1X_2^2+\left(X_3t^2+X_4^2t\right)+X_1Y_2(t)^2+3X_5Y_1(t)\leq 0$ where both $Y_1(t)$ and $Y_2(t)$ are Gaussian processes, $X_1$ and $X_3$ are normal random variables, and $X_2$, $X_4$ and $X_5$ are non-normal random variables. The set **W** may include $X_1$, $X_2$, $X_4$, $X_5$ and all standard normal variables representing the process $Y_2(t)$; i.e., $\mathbf{W}=\begin{bmatrix}X_1 & X_2 & X_4 & X_5 & Y_2(t)\end{bmatrix}$. For every realization of **W** in Eq. (7), the limit-state function for calculating $P\left(F/_{\mathbf{W}}\right)$ becomes therefore, $g(\cdot,t)=x_1^3+2x_1x_2^2+\left(X_3t^2+x_4^2t\right)+x_1y_2(t)^2+3x_5Y_1(t)$ where all lower case *x* quantities are constants and $y_2(t)$ is a deterministic function of time.

The definition of **W** is not unique. For this hypothetical example, we may also choose $\mathbf{W}=\begin{bmatrix}X_1 & X_2 & X_3 & X_4 & X_5 & Y_2(t)\end{bmatrix}$. A simple way to form **W** is to include in it all non-normal random variables of the limit state.

After the set **W** is defined, we calculate the time-dependent conditional probability $P\left(F/_{\mathbf{W}}\right)$ for each realization **w** of **W** using the concept of composite limit state (Singh, Mourelatos and Li, 2010a; Singh and Mourelatos, 2010).

For the above hypothetical example, if $\mathbf{W}=\begin{bmatrix}X_1 & X_2 & X_3 & X_4 & X_5 & Y_2(t)\end{bmatrix}$ the limit-state function becomes $g(\cdot,t)=x_1^3+2x_1x_2^2+\left(x_3t^2+x_4^2t\right)+x_1y_2(t)^2+3x_5Y_1(t)=c_a(\mathbf{w},t)+c_b(\mathbf{w})Y_1(t)$ where $c_a(\mathbf{w},t)$ is a function of time and $c_b(\mathbf{w})$ is a constant. If we thus use a spectral decomposition (KL expansion) such as in Eq. (6) to represent $Y_1(t)$ the limit-state function is a linear combination of standard normal variables for each time *t*.

If possible, we choose **W** so that for each realization **w**, the time-dependent limit state function is of the form

$$g(\mathbf{Z},\mathbf{w},t)=c_0(\mathbf{w},t)+\sum_{i=1}^{p}c_i(\mathbf{w},t)\,Z_i=c_0(\mathbf{w},t)+\mathbf{C}(\mathbf{w},t)\mathbf{Z} \tag{8a}$$

where $c_0(\mathbf{w},t)$ and $\mathbf{C}(\mathbf{w},t)$ are functions of **w** (vector of constants) and time, and $\mathbf{Z}$ is a vector of *p* standard normal random variables. In this case, each instantaneous limit state is linear in the standard normal space. If this is not possible, Eq. (8a) can be of the following more general form

$$g(\mathbf{Z},\mathbf{w},t) = c_0(\mathbf{w},t) + \sum_{i=1}^{p} c_i(\mathbf{w},t)\, Z_i = c_0(\mathbf{w},t) + \mathbf{C}(\mathbf{w},t)\, f(\mathbf{Z}) \tag{8b}$$

where $f(\mathbf{Z})$ is a function of normal variables $\mathbf{Z}$. In the remaining we will use Eq. (8a) for simplicity.

For $t_i = (i-1)\Delta t,\ i = 1,\ldots,N$, the time-dependent conditional probability $P\left(F/\mathbf{W}\right)$ is given by

$$P\left(F/\mathbf{W}\right) = P\left\{\bigcup_{i=0}^{N}\left(g(\mathbf{Z},\mathbf{w},t_i) \le 0\right)\right\}. \tag{9}$$

The composite limit state is defined by the boundary of the domain $\bigcup_{i=0}^{N}\overline{E}_i$ where the set $E_i = \{\mathbf{z} \in \mathbf{Z},\ g(\mathbf{Z},\mathbf{w},t_i) \le 0\}$ represents the instantaneous failure region at time $t_i = (i-1)\Delta t$ and $\overline{E}_i$ is the complement of $E_i$. This is similar to defining the boundary of the feasible domain for an optimization problem. Fig. 1 shows a hypothetical composite limit state function defined by a set of instantaneous limit states.

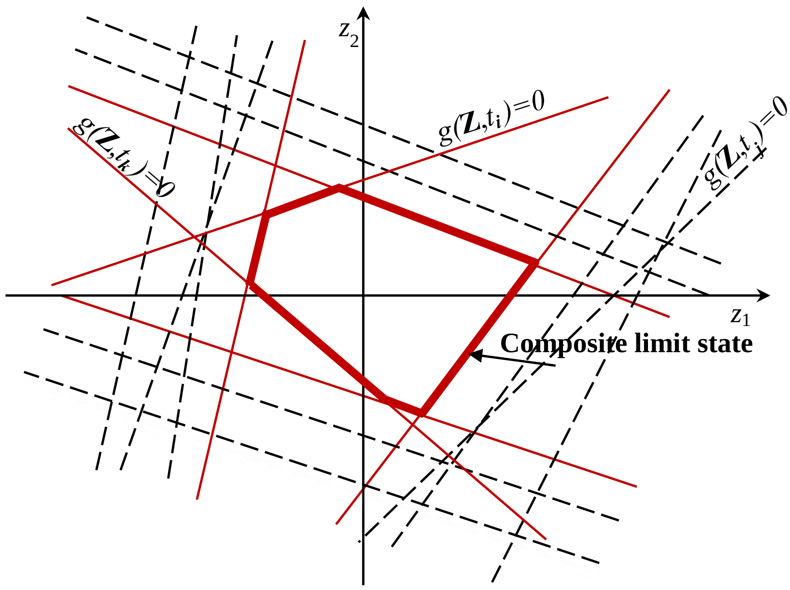


*Figure 1.* Schematic of a composite limit state

Based on this definition, the composite limit state is formed by the instantaneous limit state corresponding to the global minimum of $\beta(\mathbf{w},t)$ and a few others with $\beta(\mathbf{w},t)$ close to the global minimum. A two-step process is provided in this section to automatically identify the composite limit state.

From Eq. (8a), the exact value of the reliability index at each time $t \in [0, T]$ is

$$\beta(\mathbf{w},t)=\frac{c_0(\mathbf{w},t)}{\sqrt{\mathbf{C}(\mathbf{w},t)\mathbf{C}(\mathbf{w},t)^T}}=\frac{c_0(\mathbf{w},t)}{\sqrt{\sum_{i=1}^{p}c_i(\mathbf{w},t)^2}}\,. \tag{10}$$

The corresponding MPP vectors $\mathbf{z}^*(\mathbf{w},t)$ are

$$\mathbf{z}^*(\mathbf{w},t)=\boldsymbol{\alpha}(\mathbf{w},t)\beta(\mathbf{w},t) \tag{11}$$

where $\boldsymbol{\alpha}(\mathbf{w},t)$ is the probabilistic sensitivity vector

$$\boldsymbol{\alpha}(\mathbf{w},t)=-\frac{\mathbf{C}(\mathbf{w},t)}{\sqrt{\mathbf{C}(\mathbf{w},t)\mathbf{C}(\mathbf{w},t)^T}}\,. \tag{12}$$

In the remaining, we will drop $\mathbf{w}$ from $\beta(\mathbf{w},t)$, $\mathbf{z}^*(\mathbf{w},t)$ and $\boldsymbol{\alpha}(\mathbf{w},t)$ for simplicity.

2.3.1. *Identification of Composite Limit State*

The process to identify the composite limit states using all instantaneous limit states consists of the following two steps.

**Step 1**: Delete highly correlated instantaneous limit states

We calculate the correlation coefficient matrix $[\rho]=\lfloor\rho_{ij}\rfloor$ among all instantaneous limit states at $t_i=(i-1)\Delta t,\ i=1,\ldots,N$ where

$$\rho_{ij}=\frac{\mathbf{z}_i^*}{\left\|\mathbf{z}_i^*\right\|}\cdot\frac{\mathbf{z}_j^*}{\left\|\mathbf{z}_j^*\right\|},\ i=1,\ldots,N,\ j=i,\ldots,N\,. \tag{13}$$

The limit state functions $g(\mathbf{Z},t_i)=0$ and $g(\mathbf{Z},t_j)=0$ are assumed highly positively correlated if $\rho_{ij}\geq+0.99$. In this case if $\beta(t_i)<\beta(t_j)$, the $j^{\text{th}}$ limit state is eliminated because the $i^{\text{th}}$ limit state dominates it probabilistically; i.e., if we have failure with respect to the $i^{\text{th}}$ limit state we also have failure with respect to the $j^{\text{th}}$ limit state with a very high probability.

Let $G_1$ be the set of limit states after the dominated limit states have been deleted. The number of limit states in $G_1$ is denoted by $N_{G_1}$.

**Step 2:** Delete instantaneous limit states that are not part of the composite limit state

An instantaneous limit state $g(\mathbf{Z},t_i)=0$ among the remaining $N_{G_1}$ limit states from step 1 is deleted if the set $\left\{\mathbf{Z}: g(\mathbf{Z},t_i)=0 \;\; and \;\; \bigcap_{j=1,\, j\neq i}^{N_{G_1}} g\left(\mathbf{Z},t_j\right)\geq 0\right\}$ is null; i.e., there is no point on $g(\mathbf{Z},t_i)=0$ which belongs to the safe region defined by the remaining instantaneous limit states. This can be checked by solving a simple linear program which minimizes an arbitrary linear function such as $\sum_{i=1}^{p} z_i$ with $g(\mathbf{Z},t_i)=0$ as an equality constraint and the other limit states as inequality constraints $g\left(\mathbf{Z},t_j\right)\geq 0,\; j=1,\ldots,N_{G_1},\; j\neq i$. If no feasible solution exists the $i^{th}$ instantaneous limit state is deleted from $G_1$. This step is repeated for all $N_{G_1}$ limit states. Let the set of remaining limit states be $G_2$. The number of limit states in $G_2$ is denoted by $N_{G_2}$.

At the end of Step 2 the composite limit state is defined as a convex (not necessarily closed) polytope with $N_{G_2}$ edges.

### 2.3.2. *Calculation of Time-dependent Probability of Failure*

The time-dependent probability of failure for the set of $l=N_{G_2}$ limit states, forming the composite limit state, is given by:

$$P\left(\bigcup_{i=1}^{l} E_i\right) = \sum_{i\leq l} P(E_i) - \sum_{i_1<i_2} P\left(E_{i_1}\cap E_{i_2}\right) + \sum_{i_1<i_2<i_3} P\left(E_{i_1}\cap E_{i_2}\cap E_{i_3}\right) - \cdots + (-1)^l \sum_{i_1<i_2\ldots<i_{l-1}} P\left(E_{i_1}\cap\ldots\cap E_{i_{l-1}}\right) + (-1)^{l+1} P(E_1\cap E_2\cap\ldots\cap E_l) \tag{14}$$

where the set $E_i=\{\mathbf{z}\in\mathbf{Z},\, g(\mathbf{z},t_i)\leq 0\}$ represents the failure region corresponding to the $i^{th}$ instantaneous limit state of the composite.

The individual probabilities of failure $P(E_i)$ in Eq. (14) are equal to $\Phi(-\beta_i)$. The intersection probability $P_{ij}^f = P\left(E_i\cap E_j\right),\; i,j=1,\ldots,l$ between the $i^{\text{th}}$ and $j^{\text{th}}$ instantaneous limit states of the composite is also calculated exactly. If $\rho_{ij}=-1$, the two limit states are negative fully correlated and $p_{ij}^f=0$. If $\rho_{ij}=0$ the two limit states are uncorrelated and $P_{ij}^f=\Phi(-\beta(t_i))\Phi\left(-\beta\left(t_j\right)\right)$. If $\rho_{ij}>0$ the probability $p_{ij}^f$ is given by (Liang, Mourelatos and Nikolaidis, 2007)

$$P_{ij}^f = \Phi(-\beta(t_i))\Phi\left(-\beta\left(t_j\right)\right) + \int_0^{\rho_{ij}} \varphi\left(-\beta(t_i),-\beta\left(t_j\right);z\right)dz \tag{15}$$

where $\varphi(,;\rho)$ is the PDF of a bivariate normal vector with zero means, unit variances and a correlation coefficient $\rho$ given by

$$\varphi(-\beta_i, -\beta_j; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\frac{\beta_i^2 + \beta_j^2 - 2\rho\beta_i\beta_j}{1-\rho^2}\right] \tag{16}$$

Below, we propose a methodology which calculates the time-dependent probability of failure $P(F) = P\left(\bigcup_{i=1}^{l} E_i\right)$ exactly if the instantaneous limit states in the composite are linear. The method uses only the individual probabilities of failure $P(E_i)$ and the intersection probabilities $P_{ij}^f = P(E_i \cap E_j)$, $i, j = 1,\ldots,l$.

Using all instantaneous limit states in the composite, we identify all triangular regions (e.g., region ABG in Fig. 2), using all combinations of three different limit states. For limit states $g_i$, $g_j$ and $g_k$ with MPPs $\mathbf{z}_i^*$, $\mathbf{z}_j^*$ and $\mathbf{z}_k^*$, we test if the vector $\mathbf{z}_i^*$ is a positive linear combination of $\mathbf{z}_j^*$ and $\mathbf{z}_k^*$. This is true if there exist two positive constants $\alpha_j$ and $\alpha_k$ such that

$$\alpha_j \mathbf{z}_j^* + \alpha_k \mathbf{z}_k^* = \mathbf{z}_i^*, \qquad \alpha_j, \alpha_k > 0.$$

In this case, the limit state $g_i$ forms a triangular region with limit states $g_j$ and $g_k$. The region ABG in Fig. 2 provides an example.

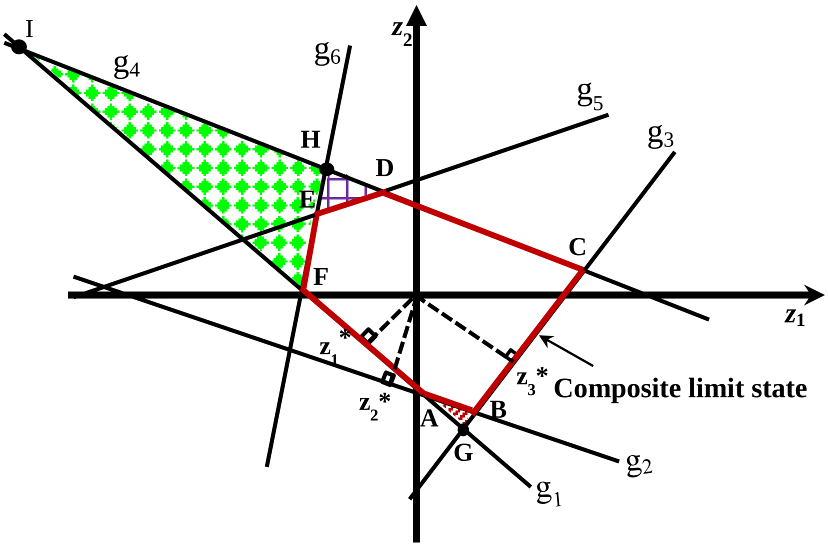


*Figure 2.* Illustration of probability of failure calculation

The probability of failure corresponding to the ABG triangular region is equal to

$$P_{ABG}^{f} = P(g_1 \geq 0,\ g_2 \leq 0,\ g_3 \geq 0) = \Phi(-\beta_2) - P_{12}^{f} - P_{23}^{f} + P_{13}^{f} \quad (17)$$

where $P_{ij}^{f}$ is calculated exactly using Eq. (15). Therefore, the time-dependent probability of failure is

$$P(F) = 1 - P_{ABCDEFA} = (1 - P_{GCDEFG}) + P_{ABG}^{f} \quad (18)$$

where $P_{ABCDEFA}$ is the probability of being within the ABCDEFA safe region of the composite limit state. Eq. (18) indicates that a new composite safe region *GCDEFG* is formed by eliminating $g_2$.

The process continues so that

$$P(F) = 1 - P_{ABCDEFA} = (1 - P_{GCDEFG}) + P_{ABG}^{f} = (1 - P_{GCHFG}) + P_{ABG}^{f} + P_{DHE}^{f} = (1 - P_{GCI}) + P_{ABG}^{f} + P_{DHE}^{f} + P_{FHI}^{f} \quad (19)$$

where

$$1 - P_{GCI} = \Phi(-\beta_3) + \Phi(-\beta_4) + \Phi(-\beta_1) - P_{34}^{f} - P_{41}^{f} - P_{13}^{f} \quad (20)$$

is the probability of being outside the safe region GCI.

The above process enlarges the safe region progressively by eliminating instantaneous limit states of the composite until we get the final three-edged convex polytope. Note that because the $P(F)$ calculation of Eq. (19) uses the exact joint probability calculation of Eq. (16), it is exact itself.

### 2.4. Summary of the Proposed Method

The following steps summarize the description of the proposed method of Sections 2.1, 2.2, and 2.3.

**Step 1**. Decompose all Gaussian random processes in the time-dependent limit state function using the spectral decomposition (K-L expansion) of Eq. (6).
**Step 2**. Choose set **W** for the total probability theorem (Eq. 7) so that for each realization of **W** the limit state becomes preferably a linear combination of normal random variables at each time *t*.
For a realization **w** of the chosen set **W** perform steps 3 through 6:
**Step 3**. Express the time-dependent limit-state function as a function (preferably linear) of normal random variables as in Eq. (8a) or (8b). This step constitutes a function evaluation.
**Step 4**. For the time of interest $t \in [0, T]$ calculate the reliability index and correlation coefficient matrix for all discrete times from zero to *t* using Equations (10) through (13) or a FORM approach if Eq. (8b) is used.
**Step 5**. Identify the composite limit state using the proposed two-step composite limit state identification process and the information from Step 4.
**Step 6**. Obtain the correlation coefficient matrix for the *l* instantaneous limit states in the composite using Eq. (13). Calculate the conditional probability $P\left(F/\mathbf{W}\right)$ using the proposed approach of Eqs (14) through (20).

**Step 7**. Obtain the time-dependent probability of failure at time *t* by evaluating the total probability theorem integral of Eq. (7) using numerical integration, simple MCS or importance sampling. $P\left(F/\mathbf{W}\right)$ may be calculated directly or using a pre-built metamodel as illustrated in the example below.

## 3. Example: Design of a Hydrokinetic Turbine Blade

This example is adopted from (Hu and Du, 2013). It involves the design of a hydrokinetic turbine blade under time-dependent river flow load. Figures 3 and 4 from (Hu and Du, 2013) show a simplified cross-section of the blade, and the blade under river flow load, respectively.

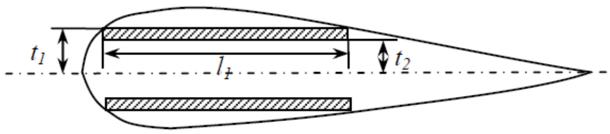


*Figure 3.* Cross-section of turbine blade at the root (Hu and Du, 2013)

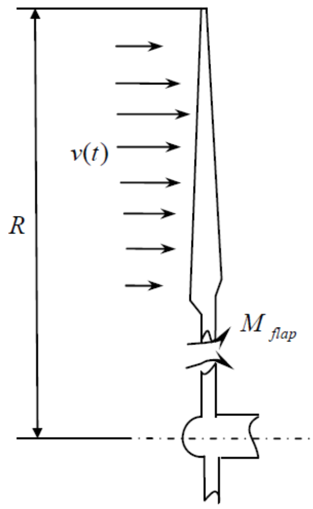


*Figure 4.* River flow load on the turbine blade (Hu and Du, 2013)

The flapwise bending moment $M_{flap}$ at the root of the blade is given by (Hu and Du, 2013; Martin, 2008)

$$M_{flap} = \frac{1}{2} \rho\, v(t)^2 C_m \tag{21}$$

where $v(t)$ is the random river flow velocity, $\rho = 10^3\ kg/m^3$ is the river flow density, and $C_m = 0.3422$ is the coefficient of moment provided by the blade element momentum theory.

Based on (Hu and Du, 2013), the monthly river velocity $v(t)$ is modeled by a narrowband, non-stationary Gaussian process with mean $\mu_v(t)$, standard deviation $\sigma_v(t)$, and autocorrelation function $\rho_v(t_1,t_2)$ given by

$$\mu_v(t)=\sum_{i=1}^{4}\alpha_i^m \sin\left(b_i^m t + c_i^m\right) , \tag{22}$$

$$\sigma_v(t)=\sum_{j=1}^{4}\alpha_j^s \exp\left\{-\left[\left(t-b_j^s\right)/c_j^s\right]^2\right\} \tag{23}$$

and

$$\rho_v(t_1,t_2)=\cos\left(2\pi\left(t_2-t_1\right)\right) \tag{24}$$

where the constants *a*, *b* and *c* for Eq. (22) are

$$\begin{aligned}
&a_1 = 3.815, && b_1 = 0.2895, && c_1 = -0.2668\\
&a_2 = 2.528, && b_2 = 0.5887, && c_2 = 0.9651\\
&a_3 = 1.176, && b_3 = 0.7619, && c_3 = 3.116\\
&a_4 = -0.07856, && b_4 = 2.183, && c_4 = -3.161
\end{aligned}$$

and the constants for Eq. (23) are

$$\begin{aligned}
&a_1 = 0.7382, && b_1 = 6.456, && c_1 = 0.9193\\
&a_2 = 1.013, && b_2 = 4.075, && c_2 = 1.561\\
&a_3 = 1.875, && b_3 = 9.913, && c_3 = 6.959\\
&a_4 = 1.283, && b_4 = 1.035, && c_4 = 2.237
\end{aligned}$$

The limit state function is

$$g(\cdot)=\varepsilon_a-\frac{M_{flap}\,t_1}{E\,I}=\varepsilon_a-\frac{\rho\, v(t)^2 C_m\, t_1}{2\,E\,I} \tag{25}$$

where $\varepsilon_a$ is the allowable strain of the material, $E$ = 14 GPa is the Young's modulus, and $I=\frac{2}{3}l_1\left(t_1^3-t_2^3\right)$ is the area moment of inertia at the root of the blade. The dimension variables $l_1$, $t_1$ and $t_2$ are shown in Fig. 3. Table I provides the distribution information of the random variables and velocity random process.

| Table I. Distribution information for hydrokinetic turbine blade example | | | | |
|---|---|---|---|---|
| **Variable** | **Mean** | **Std** | **Distribution** | **Auto Cor.** |
| v (m/s) | $\mu_v(t)$ | $\sigma_v(t)$ | GP | Eq. 24 |
| $l_1$ | 0.22 | 0.0022 | Gaussian | N/A |
| $t_1$ | 0.025 | 0.00025 | Gaussian | N/A |
| $t_2$ | 0.019 | 0.00019 | Gaussian | N/A |
| $\varepsilon_{allow}$ | 0.025 | 0.00025 | Gaussian | N/A |

The time interval $[0, 12]$ months was discretized uniformly using $\Delta t = 0.2$ months. The covariance matrix $\mathbf{\Sigma}$ was formed and its eigenvalues and corresponding eigenvectors were calculated. Because of the periodicity of the autocorrelation function (Eq. 24), only the first two eigenvalues $\lambda_1 = 110.16$ and $\lambda_2 = 107.96$ were non-zero. Therefore, the spectral decomposition of Eq. (6) for $v(t)$ has only two terms besides the mean; i.e.,

$$v(t) = \mu_v(t) + \sqrt{\lambda_1} \cdot \mathbf{\Phi}_1^T(t) \cdot Z_1 + \sqrt{\lambda_2} \cdot \mathbf{\Phi}_2^T(t) \cdot Z_2 . \tag{26}$$

Fig. 5 shows realizations (trajectories) of $v(t)$. We observe that many realizations of $v(t)$ exhibit negative values for some instances in the time of interest $[0, 12]$ months.

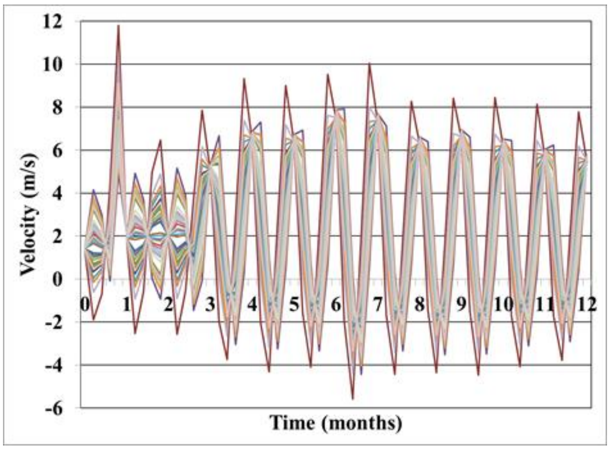


*Figure 5.* Realizations of $v(t)$

The limit state function of Eq. (25) is re-written as

$$g(\cdot) = C - v(t)^2 \leq 0 \tag{27}$$

where $C = \dfrac{2EI\varepsilon_a}{\rho C_m t_1}$ since $E$, $I$, $\rho$, $C_m$ and $t_1$ are positive quantities. It can be shown that

$$P\left(F/_{\mathbf{X}}\right) = P\left[C - v(t)^2 \le 0\right] = 2P\left[\sqrt{C} - v(t) \le 0\right] + 2P\left[v(t) + \sqrt{C} \le 0\right] \tag{28a}$$

Considering that the reliability index of $v(t) + \sqrt{C}$ is much higher than that of $\sqrt{C} - v(t)$ at all times, Eq. (28a) is approximated as

$$P\left(F/_{\mathbf{X}}\right) = P\left[C - v(t)^2 \le 0\right] \approx 2P\left[\sqrt{C} - v(t) \le 0\right]. \tag{28b}$$

We will now calculate $P\left[\sqrt{C} - v(t) \le 0\right]$ and use it in Eq. (28b) to obtain $P\left(F/_{\mathbf{X}}\right) = P\left[C - v(t)^2 \le 0\right]$. The reliability index at each time $t$ is

$$\beta(\mathbf{w}, t) = \frac{\sqrt{C(\mathbf{w})} - \mu_v(t)}{\sqrt{d_1(t)^2 + d_2(t)^2}} \tag{29}$$

where $d_1(t) = \sqrt{\lambda_1} \cdot \mathbf{\Phi}_1^T(t)$ and $d_2(t) = \sqrt{\lambda_2} \cdot \mathbf{\Phi}_2^T(t)$ . Fig. 6 shows $\beta(t)$ when all random variables in $C$ are at their means.

Because $C$ is only a function of **w** in Eq. (29), the time-dependent curve of $\beta$ has the same shape for different values of **w**. It simply moves up or down as **w** changes. Thus, the shape and composition of the composite limit state at each time $t$ is the same for different **w**'s although it may shrink or expand. This is also supported by the fact that the probabilistic sensitivity vector

$$\boldsymbol{\alpha}(\mathbf{w}, t) = + \frac{1}{\sqrt{d_1(t)^2 + d_2(t)^2}} \tag{30}$$

does not depend on **w**.

Hence for each **w**, the composition of the composite limit state is the same and the correlation coefficient matrix among its instantaneous limit states is also the same. Fig. 7 shows the composite limit state for $t$ = 8.2 months when all random variables are at their means. The MPP vectors are

$$\mathbf{z}^*(\mathbf{w}, t) = \frac{\sqrt{C(\mathbf{w})} - \mu_v(t)}{d_1(t)^2 + d_2(t)^2}\,. \tag{31}$$

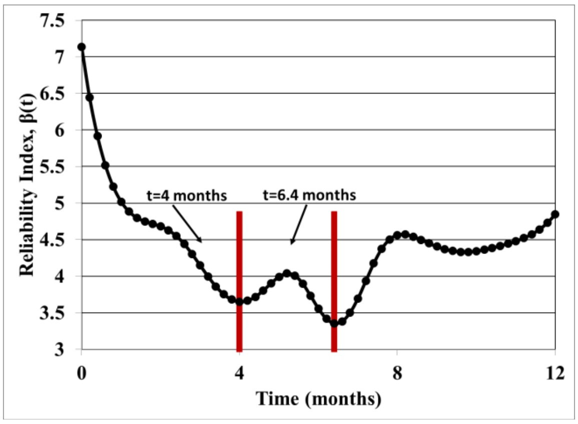


*Figure 6.* Reliability index with random variables at mean values

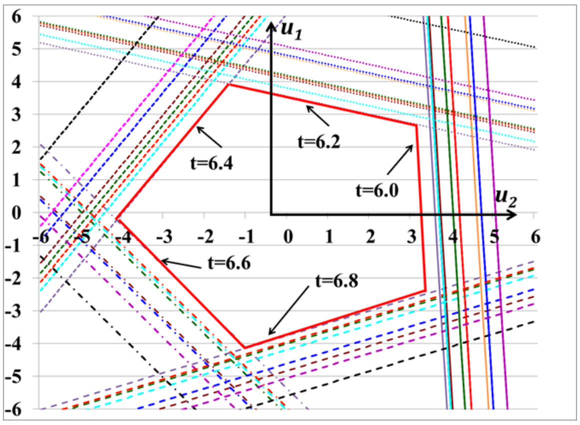


*Figure 7.* Composite limit state for $t$ = 8.2 months with random variables at mean values

As we have mentioned, the composite limit state is formed by the instantaneous limit state corresponding to the global minimum of $\beta(\mathbf{w},t)$ and a few others with $\beta(\mathbf{w},t)$ close to the global minimum. According to Fig. 6, the minimum reliability index between 0 and 8.2 months is 3.4 and occurs at 6.4 months. Therefore, the instantaneous limit state at 6.4 months is part of the composite limit state. The remaining instantaneous limit states in the composite are identified using the two-step process of Section 2.3.

Fig. 8 shows the conditional probability of failure at $t$ = 8.2 months for different values of the random variables $\mathbf{W} = [\varepsilon_a \quad t_1 \quad l_1 \quad t_2]$. Each line represents the change in the probability of failure when three of the random variables are at their mean values while the random variable of interest varies between $\mu - 3.5\sigma$ and $\mu + 3.5\sigma$ at increments of one $\sigma$. We observe that $P\left(F/_{\mathbf{W}}\right)$ is most sensitive (largest variation from

$\mu - 3.5\sigma$ to $\mu + 3.5\sigma$) to $t_1$ followed by $t_2$ . $P\left(F/\mathbf{W}\right)$ also increases with decreasing $t_1$, $l_1$ and $\varepsilon_a$ and decreases with increasing $t_2$.

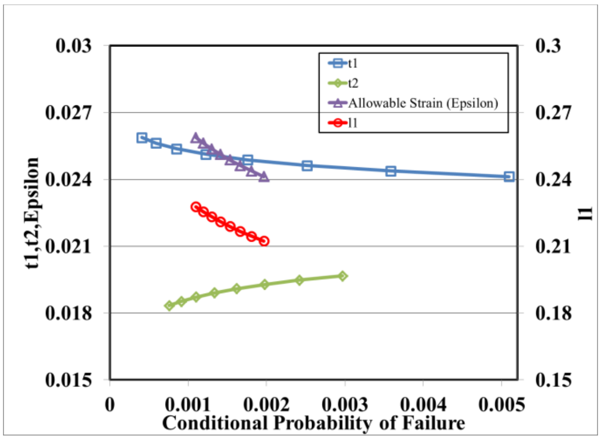


*Figure 8.* Sensitivity of conditional probability of failure at $t$ = 8.2 with $t_1$, $t_2$, $\varepsilon_a$, and $l_1$

The integral of Eq. (7) indicates that $P_f(0,T)$ is the mean value of $P\left(F/\mathbf{W}\right)$. To calculate this mean value, we built a Kriging model of $P\left(F/\mathbf{W}\right)$ and used it to obtain the average of 5,000 replications.

For the Kriging model, we used a rectangular domain to define the training points. Each side of the domain corresponds to a random variable in $\mathbf{W} = [\varepsilon_a \quad t_1 \quad l_1 \quad t_2]$, ranging from $\mu - 3.5\sigma$ to $\mu + 3.5\sigma$ where $\mu$ and $\sigma$ are the mean and standard deviation of the variable. We then used a regular grid with either three or five points along each random variable resulting in a total of 81 or 625 training points respectively, and calculated $P\left(F/\mathbf{W}\right)$ at each training point. The Kriging model was based on a second order polynomial regression and a Gaussian correlation structure.

Fig. 9 shows the value of $P\left(F/\mathbf{W}\right)$ at $t$ = 8.2 months for the 81 training points. The probability increases if $t_1$, $l_1$ and $\varepsilon_a$ approach their lowest values $\mu - 3.5\sigma$ and $t_2$ approaches its highest value of $\mu + 3.5\sigma$. Fig. 10 shows the conditional probability of failure $P\left(F/\mathbf{W}\right)$ through time for the training points of Table II.

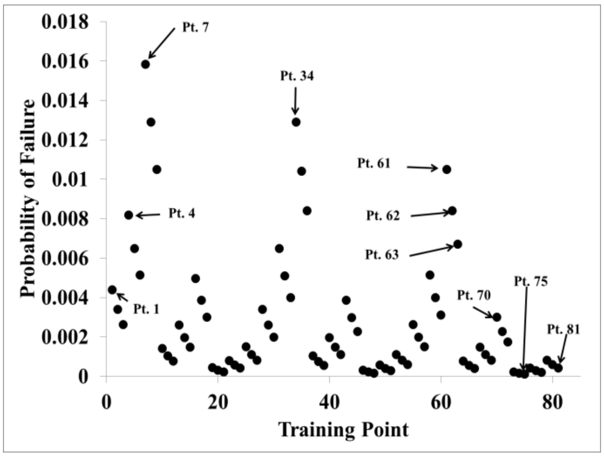


*Figure 9.* Conditional probability of failure at $t$ = 8.2 at 81 Kriging training points

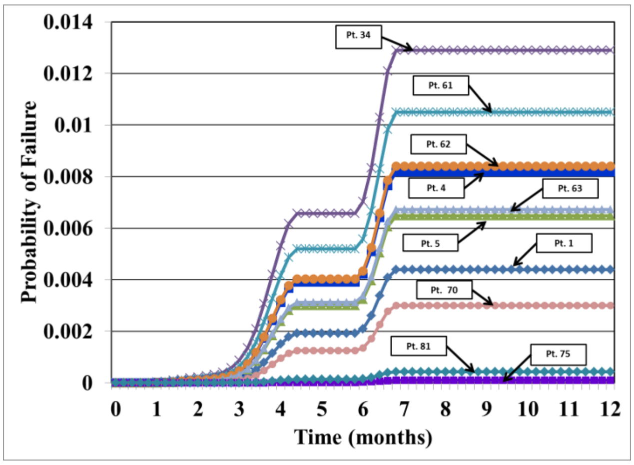


*Figure 10.* Conditional probability of failure at selected Kriging training points

| Table II. Definition of training points in Fig. 10 | | | | |
|---|---|---|---|---|
| Training Point | Epsilon | t1 (m) | t2 (m) | l1 (m) |
| 1 | 0.024125 | 0.024125 | 0.018335 | 0.2123 |
| 4 | 0.024125 | 0.024125 | 0.019 | 0.2123 |
| 5 | 0.024125 | 0.024125 | 0.019 | 0.22 |
| 34 | 0.025 | 0.024125 | 0.019665 | 0.2123 |
| 61 | 0.025875 | 0.024125 | 0.019665 | 0.2123 |
| 62 | 0.025875 | 0.024125 | 0.019665 | 0.22 |
| 63 | 0.025875 | 0.024125 | 0.019665 | 0.2277 |
| 70 | 0.025875 | 0.025 | 0.019665 | 0.2123 |
| 75 | 0.025875 | 0.025875 | 0.018335 | 0.2277 |
| 81 | 0.025875 | 0.025875 | 0.019665 | 0.2277 |

Fig. 11 compares the estimated time-dependent $P_f\left(0, T\right)$ using the Kriging model of $P\left(F/_{\mathbf{W}}\right)$ with 81 and 625 training points, with that obtained by MCS with 1,000,000 replications. The accuracy with 81 training points, and therefore function evaluations (see step 3 of Section 2.4), is very good. It improves drastically however, with 625 training points. This is strong evidence that building the Kriging model sequentially placing points where the variance of the Kriging estimate is large, will improve the accuracy and efficiency of our approach considerably. We plan to do so in future work. We also plan to use adaptive importance sampling.

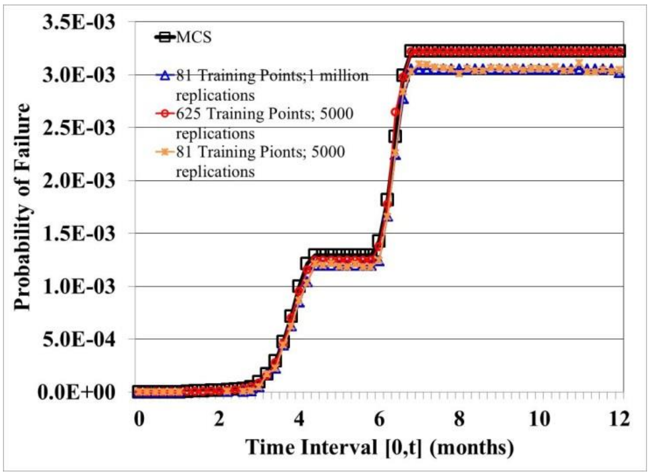


*Figure 11.* Comparison of $P_f\left(0,T\right)$ between the proposed method and MCS

## 4. Summary, Concluations and Future Work

We proposed a new reliability analysis method for time-dependent problems with limit-state functions of input random variables, input random processes which may depend on time explicitly or implicitly, using the total probability theorem and the concept of composite limit state. The total probability theorem calculates the time-dependent probability of failure using a time-dependent conditional probability which is computed accurately and efficiently in the standard normal space using FORM and a composite limit state of linear instantaneous limit states. The integral expressing the total probability theorem can be calculated using numerical integration such as Gauss quadrature, for low-dimensional problems with a small number of random variables. Otherwise, we can use simple MCS or adaptive importance sampling based on a pre-built Kriging metamodel of the conditional probability. The latter is needed for efficiency purposes. The design of a hydrokinetic turbine blade example under time-dependent river flow load demonstrated the advantages of the proposed approach and identified areas for improvement.

Future work will mainly concentrate on the efficient and accurate evaluation of the integral in the total probability theorem of Eq. (7) using a simulation-based approach for high dimensional problems based on adaptive importance sampling. Emphasis will also be placed on building an accurate metamodel of the conditional probability of failure using an adaptive Kriging approach in conjunction with time-dependent metamodeling as in (Wehrwein and Mourelatos, 2009).

## 5. Acknowledgement

We would like to acknowledge the technical and financial support of the Automotive Research Center (ARC) in accordance with Cooperative Agreement W56HZV-04-2-0001 U.S. Army Tank Automotive Research, Development and Engineering Center (TARDEC) Warren, MI.

## 6. References

1. Abramowitz, M. and Stegun, I. A., eds, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover Publications, New York, 1965.
2. Andrieu-Renaud, C., Sudret, B. and Lemaire, M. The PHI2 Method: A Way to Compute Time-Variant Reliability. *Reliability Engineering and Safety System*, 84(1):75-86, 2004.
3. Au, S. K. and Beck, J. L. Estimation of Small Failure Probability in High Dimensions Simulation. *Probabilistic Engineering Mechanics*, 16:263-277, 2001.
4. Au, S. K. and Beck, J. L. Subset Simulation and its Application to Seismic Risk Based on Dynamic Analysis. *Journal of Engineering Mechanics*, 129:901-917, 2003.
5. Beck, J. L. and Au, S. K. Bayesian Updating of Structural Models and Reliability Using Markov Chain Monte Carlo Simulation. *Journal of Engineering Mechanics*, 128(4), 380-391, 2002.
6. Chen, J. B. and Li, J. The Extreme Value Distribution and Dynamic Reliability Analysis of Nonlinear Structures with Uncertain Parameters. *Structural Safety*, 29(2), 77-93, 2007.

7. Ching, J., Beck, J. L. and Au, S. K. Reliability Estimation for Dynamic Systems Subject to Stochastic Excitation Using Subset Simulation with Splitting. *Computer Methods in Applied Mechanics and Engineering*, 194, 1557-1579, 2005.
8. Hu, Z. and Du, X. Reliability Analysis for Hydrokinetic Turbine Blades. *Renewable Energy*, 48, 251-262, 2012.
9. Hu, Z. and Du, X. A Sampling Approach to Extreme Value Distribution for Time-dependent Reliability Analysis. *ASME Journal of Mechanical Design*, 135(7):071003 (8 pages), 2013.
10. Hu, Z., Li, H., Du, X. and Chandrashekhara, K. Simulation-based Time-dependent Reliability Analysis for Composite Hydrokinetic Turbine Blades. *Structural and Multidisciplinary Optimization*, 47(5):765-781, 2013.
11. Li, J. and Mourelatos, Z. P. Time-Dependent Reliability Estimation for Dynamic Problems using a Niching Genetic Algorithm. *ASME Journal of Mechanical Design*, 131(7):071009 (13 pages), 2009.
12. Li, C. C. and Kiureghian, A. D. Optimal Discretization of Random Fields. *Journal of Engineering Mechanics*, 119(6), 1136-1154, 1993.
13. Liang, J., Mourelatos, Z. P. and Nikolaidis, E. A Single-Loop Approach for System Reliability-Based Design Optimization. *ASME Journal of Mechanical Design*, 129(12):1215-1224, 2007.
14. Madsen, P. H. and Krenk, S. An Integral Equation Method for the First Passage Problem in Random Vibration. *Journal of Applied Mechanics*, 51:674-679, 1984.
15. Martin, O. L. H. *Aerodynamics of Wind Turbines, 2nd Edition*. Earthscan, Sterling, VA, 2008.
16. Rackwitz, R. Computational Techniques in Stationary and Non-Stationary Load Combination – A Review and Some Extensions. *Journal of Structural Engineering*, 25(1):1-20, 1998.
17. Rice, S. O. Mathematical Analysis of Random Noise. *Bell Syst Tech J*, 23:282–332, 1954. [Re-published in: Wax N, editor. *Selected Papers on Noise and Stochastic Processes*, New York, Dover, 1954].
18. Savage, G. J. and Son, Y. K. Dependability-Based Design Optimization of Degrading Engineering Systems. *ASME Journal of Mechanical Design*, 131(1):011002 (10 pages), 2009.
19. Singh, A. and Mourelatos, Z. P. On the Time-dependent Reliability of Non-Monotonic, Non-Repairable Systems. *SAE International Journal of Materials and Manufacturing*, 3(1):425-444, 2010.
20. Singh, A. and Mourelatos, Z. P. Time-Dependent Reliability Estimation for Dynamic Systems Using a Random Process Approach. *SAE International Journal of Materials & Manufacturing*, 3(1):339-355, 2010.
21. Singh, A., Mourelatos, Z. P. and Li, J. Design for Lifecycle Cost using Time-dependent Reliability. *ASME Journal of Mechanical Design*, 132(9):091008 (11 pages), 2010a.
22. Singh, A., Mourelatos, Z. P. and Li, J. Design for Lifecycle Cost and Preventive Maintenance Using Time-Dependent Reliability. *Advanced Materials Research*, 118-120:10-16, 2010b.
23. Singh, A., Mourelatos, Z. P. and Nikolaidis, E. Time-Dependent Reliability of Random Dynamic Systems using Time-Series Modeling and Importance Sampling. *SAE International Journal of Materials & Manufacturing*, 4(1):929-946, 2011.
24. Son, Y. K. and Savage, G. J. Set Theoretic Formulation of Performance Reliability of Multiple Response Time-Variant Systems due to Degradations in System Components. *Quality and Reliability Engineering International*, 23(2):171-188, 2007.
25. Sudret, B. Analytical Derivation of the Outcrossing Rate in Time-variant Reliability Problems. *Structure and Infrastructure Engineering*, 4(5):356-362, 2008.

26. Wang, Z., Mourelatos, Z. P., Li, J., Singh, A. and Baseski, I. Time-dependent Reliability of Dynamic Systems using Subset Simulation with Splitting over a Series of Correlated Time Intervals. In *Proceedings of ASME 2013 Design Engineering Technical Conferences*, Paper DETC2013-12257, Portland, OR, 2013.
27. Wang, Z. and Wang, P. A Nested Extreme Response Surface Approach for Time-dependent Reliability-based Design Optimization. *ASME Journal of Mechanical Design*, 134(12):121007 (14 pages), 2012.
28. Wehrwein D. and Mourelatos, Z. P. Optimization of Engine Torque Management under Uncertainty for Vehicle Driveline Clunk using Time-Dependent Metamodels. *ASME Journal of Mechanical Design*, 131(5), 2009.
29. Zhang, J. and Du, X., Time-dependent Reliability Analysis for Function Generator Mechanisms. *ASME Journal of Mechanical Design*, 133(3):031005 (9 pages), 2011.