Dynamic Analysis of Uncertain Structures Using Imprecise Probability

Mehdi Modares and Joshua Bergerson
Illinois Institute of Technology, mmodares@iit.edu & jbergers@iit.edu

Abstract: A new method for dynamic response spectrum analysis of structures with uncertainty in their mechanical properties utilizing the notion of imprecise probability is developed. This finite-element-based method is capable of obtaining probabilistic bounds of the dynamic response of the structure with uncertainty defined by enveloping p-boxes. The developed method obtains probabilistic bounds on 1) the mode shapes, 2) modal coordinates, and 3) modal participation factor, leading to the p-boxes of modal responses. Finally maximum modal responses are combined to obtain the structure's maximum total response with consideration of uncertainty. Numerical examples demonstrating the developed method are included.

Keywords: Structural Dynamics, Uncertainty, Imprecise Probability, P-boxes

1. Introduction

1.1. Background

The design of a structure to safely resist suddenly applied loads, such as earthquake or wind loads, requires dynamic analysis procedures. Conventional dynamic response spectrum analysis is one of the methods of structural dynamics that obtains the structure's total response using a combination of modal response maxima. However, one major shortcoming of traditional dynamic analyses is that these methods require deterministic parameters; i.e., they are unable to handle uncertainty in any parameter of the system. Moreover, these analytical schemes need to be performed for each change in any of the input parameters throughout the design process.

Uncertainty is categorized as either aleatoric or epistemic. Aleatoric uncertainty encompasses uncertainties due to inherent randomness throughout a system. This type of uncertainty cannot be reduced by additional information or higher precision, though it can be accounted for using traditional theories of probability. Epistemic uncertainty can be caused by incorrect modeling or assumptions due to a lack of or insufficient knowledge to properly control and accurately model the system. Epistemic uncertainty can be reduced in a system by gathering additional data to improve modeling. Possibilistic methods, including interval and fuzzy analyses, are typically utilized for handling epistemic uncertainties by bounding the uncertainties. Both probabilistic and possibilistic methods have limitations; the prior requires random variables to follow an assumed distribution while the latter is unable to consider probability distribution for bounded parameters.

Imprecise probability is a polymorphic approach of quantifying uncertainty that provides a framework for addressing the shortcomings of both traditional probabilistic and possibilistic methods for modeling uncertainty. It allows for a probabilistic modeling of possibilistic bounds. Thus, imprecise probability provides a method for modeling uncertainty that requires fewer assumptions which may significantly influence the results of the analysis.

Probability boxes are one method of modeling imprecise probability. This method sets probabilistic bounds on the Cumulative Distribution Function (CDF) which are guaranteed to bound the true CDF of the system. By bounding the CDF, no assumptions as to the form of the Probability Density Function (PDF) are required, thus addressing the shortcomings of both traditional possibilistic and probabilistic methods of modeling uncertainty, while still allowing probabilistic modeling of the CDF bounds.

In this work, a method for dynamic response analysis of structures using p-box based imprecise probability is developed. Probabilistic bounds on mode shapes, modal coordinates, and modal participation factors are all used to generate the probabilistic bounds on the modal responses. All modal contributions are combined to finally obtain the maximum total response of the system to an applied dynamic load.

2. Fundamentals of Engineering Uncertainty Analysis

2.1. IMPRECISE PROBABILITY

One of the earlier methods of modeling imprecise probability is the Dempster-Shafer (DS) method. This approach uses intervals to represent uncertainty on a deterministic CDF (Dempster 1967, Shafer 1976, 1986). Dependency bounds convolutions is a framework offered by Williamson and Downs (1990) which allows for combining imprecise probabilities for all four basic arithmetic operations, utilizing enveloping methods to retain robustness in calculations and avoid overestimation in results. Another method of modeling imprecise probabilities known as Probability Bounds Analysis (PBA) was developed by Ferson and Donald (1998). Probability boxes are another method of modeling imprecise probability by setting bounds on the CDF for an uncertain parameter and then discretizing these bounds to create probabilistic interval bounds (Ferson et al. 1998). While several methods exist for modeling imprecise probability, some have shown considerable complexities when combining multiple imprecise probabilities, such as the DS method. Ferson et al. (2003) examined several existing methods for modeling imprecise probability in order to assess the value of these methods. During this study, the authors came to the conclusion that the DS method was highly correlated to the p-box method of modeling imprecise probability, and offered procedures for converting between the two methods. The authors also examined multiple existing methods of combining imprecise probabilities, concluding that enveloping procedures were the best approach as these methods ensure the computed bounds contain all true possible results; enveloping is used for combining p-box based imprecise probabilities in this paper. A finite-element-based method of dynamic analysis with interval defined imprecise probabilities was developed by Modares et al. (2006). The method generates exact interval bounds on natural circular frequencies by utilizing the monotonic behavior of eigenvalues of symmetric matrices subjected to non-negative definite perturbations. This work was further expanded to include response spectrum analysis, entitled Interval Response Spectrum Analysis (IRSA), which obtains interval bounds on mode shapes and modal participation factors and the maximum modal coordinate (Modares and Mullen 2013). These values are used to compute the maximum modal response and the contribution of all modal responses are combined to compute the maximum total response. Moreover, a framework for handling uncertainty in applied loading in static problems was developed by Zhang et al. (2010).

2.2. P-BOX MODELING OF IMPRECISE PROBABILITY

As previously mentioned, p-boxes are a method of modeling imprecise probability by setting probabilistic bounds on the CDF that are guaranteed to bound all possible values, thus allowing modeling of both epistemic and aleatoric uncertainty simultaneously. Figure 1 below shows an example p-box, consisting of various probabilistic intervals. It is noted that the left and right curves are the upper and lower bounds on the CDF, respectively. Because the bounds are probabilistic bounds on possibility, as more information becomes available the p-box may be updated as uncertainty within the system decreases (Figure 2).

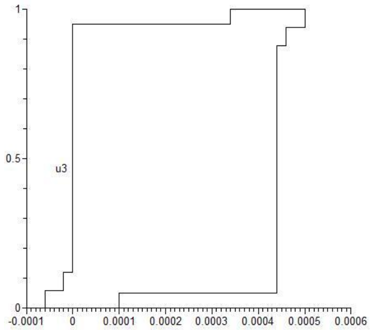

Figure 1. P-box modeling of imprecise probability (Horizontal: information intervals, Vertical: CDF).

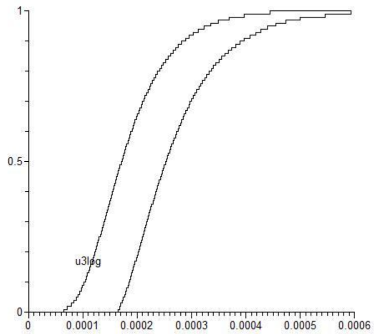

Figure 2. Additional information is used to sharpen bounds as uncertainty in the system is reduced (Horizontal: new information intervals, Vertical: CDF).

2.3. P-BOX REPRESENTATION OF INPUT UNCERTAINTY

Consider a general p-box with bounds $\overline{F_i}(x)$ and $\underline{F_i}(x)$ for an uncertain input parameter in a system (Figure 3).

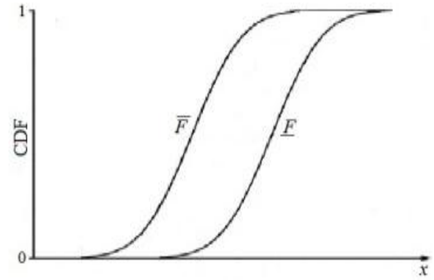

Figure 3. A general p-box.

If a vertical line is drawn within this p-box (selecting a deterministic quantity for the random variable), $\overline{F_i}(x)$ and $\underline{F_i}(x)$ are the upper and lower bounds on the CDF value for that value of the random variable (Figure 4). This notion of a probability box means that the value of the random variable is considered to be deterministic while the value of the CDF contains all uncertainties.

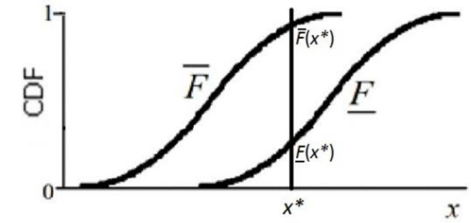

Figure 4. A vertical line drawn through the general p-box represents a deterministic value of the random variable with all uncertainty contained within the value of the CDF (vertical intervals).

If instead a horizontal line is drawn within the general p-box (selecting a deterministic quantity for the CDF), then $\overline{F_i}(x)$ and $\underline{F_i}(x)$ are the lower and upper bounds on the random variable for that value of the CDF (Figure 5). This notion of a p-box means that the value of the random variable contains all uncertainties while the value of the CDF is considered to be deterministic.

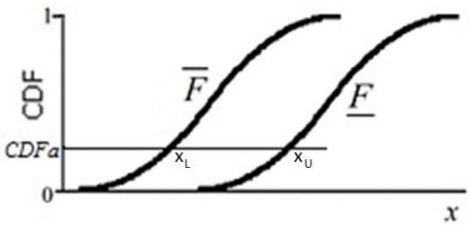

Figure 5. A horizontal line drawn through the general p-box represents a deterministic value of the CDF with all uncertainty contained within the value of the random variable (horizontal intervals).

In this work, all uncertainty existing within a given system is modeled within the value of the random variable while the CDF values are assumed to be deterministic. Thus, $\overline{F_i}(x)$ and $\underline{F_i}(x)$ are the bounding functions for the p-box describing the uncertainty existing in the value of the random variable.

A p-box is discretized for calculations into a group of intervals. The discretization is performed by considering several horizontal cuts at various CDF values. The discretization is then done by enveloping the respective bounds by selecting the minimum value of the random variable corresponding to the minimum CDF value of $\overline{F_i}(x)$ within the CDF interval and likewise selecting the maximum value of the random variable corresponding to the maximum CDF value of $\underline{F_i}(x)$ within the CDF interval.

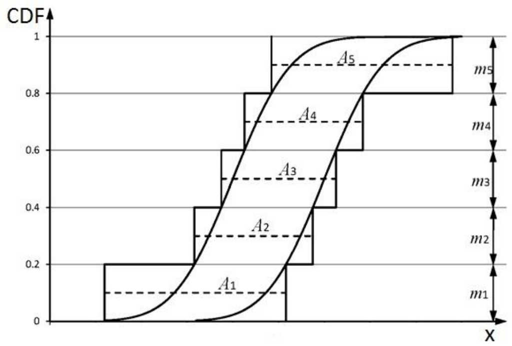

Figure 6. Illustration of the enveloping procedure for the discretization of a p-box.

In this work, probability boxes are discretized into *z* intervals, each with equal probability mass of (1/*z*). For $\gamma \in [0,1]$ representing the value of the CDF functions $\overline{F_i}(x)$ and $\underline{F_i}(x)$, the discretization is done by selecting values of random variable x corresponding to $\gamma = (q-1)/z$ and $\gamma = q/z$ for $\overline{F_i}(x)$ and $\underline{F_i}(x)$ respectively for every uncertain member i for $q = 1, 2, \ldots, z$. This is written using the inverse functions $\overline{F_i}^{-1}(x)$ and $\underline{F_i}^{-1}(x)$ as:

$$x_{iqL} = \overline{F_i}^{-1}\left(\frac{q-1}{z}\right) \tag{1}$$

$$x_{iqU} = \underline{F_i}^{-1}\left(\frac{q}{z}\right) \tag{2}$$

These bounds can be combined for each value of *q* and written in interval form for member *i* as:

$$\tilde{x}_{iq} = [x_{iqL}, x_{iqU}] \tag{3}$$

where each $\tilde{x}_{iq}$ has a probability mass of (1/*z*) for $i = 1, 2, \ldots, N$, where *N* is the number of members in the system.

3. Imprecise Probability Response Spectrum Analysis

3.1. Review of Deterministic Response Spectrum Analysis

For a dynamic system without uncertainty, the following steps comprise deterministic Response Spectrum Analysis (RSA):

1. Establish structural properties.
    a) Define mass and stiffness matrices, $[\mathrm{M}]$ and $[\mathrm{K}]$, respectively.
    b) Assume a modal damping ratio, ζ_n.
2. Solve the generalized eigenvalue problem between the mass and stiffness matrices.
    a) Compute natural circular frequencies, ω_n.
3. Compute mode shapes, $\{\varphi_n\}$.
4. Analyze modal responses.
    a) Using the excitation response spectrum, compute the maximum modal coordinate, $D_{n_{max}}$, for computed natural circular frequencies and assumed modal damping ratio.
    b) Compute the modal participation factor, Γ_n.
5. Calculate the maximum modal responses by multiplying each mode shape by its corresponding maximum modal coordinate and modal participation factor.
6. Calculate the maximum total response by combining the contributions of all maximum modal responses using appropriate combination method, such as square root sum of squares (SRSS).

3.2. Formulation of Imprecise Probability Response Spectrum Analysis

The general algorithm for Imprecise Probability Response Spectrum Analysis (IPRSA) is as follows:

For each realization *r* of the simulation:

1. Establish uncertain structural properties.
    a) Randomly selected CDF values for all independent uncertainties in the system and use these CDF values to compute interval mass and stiffness matrices, $[\widetilde{\mathrm{M}}]$ and $[\widetilde{\mathrm{K}}]$, respectively.
    b) Assume a modal damping ratio, ζ_n.
2. Determine interval bounds on the uncertain natural circular frequencies, $\widetilde{\omega}_n$.

3. Compute interval bounds for uncertain mode shapes, $\{\tilde{\varphi}_n\}$.

4. Determine modal responses.

   a) Using the excitation response spectrum, compute interval bounds on the uncertain modal coordinate, $\widetilde{D}_n$, and use this to compute the maximum modal coordinate, $D_{n_{max}}$, for computed interval natural circular frequencies and assumed modal damping ratio.
   b) Compute interval bounds on the uncertain modal participation factors, $\tilde{\Gamma}_n$.

5. Calculate the maximum uncertain maximum modal responses by multiplying each uncertain mode shape by its corresponding maximum modal coordinate and maximum uncertain modal participation factor.

6. Calculate the (conservative) maximum total response by combining all maximum uncertain maximum modal responses using a combination method, such as square root sum of squares (SRSS).

For each realization, all computed interval bounds are stored. After a sufficiently large number of realizations are completed, all upper and lower interval bounds are sorted in order to generate probability boxes for uncertain outputs and probabilistic upper-bounds on maximum uncertain maximum modal responses and maximum total response.

4. Quantification of Imprecise Parameters

4.1. Establish Uncertain Structural Properties (P-box Representation of Uncertainty)

For uncertainties existing in the stiffness properties of a structural system, a p-box must be constructed from either known or assumed parameters. If independent uncertainties in the material properties (e.g. in the modulus of elasticity of member i, E_i) are considered, then independent probability boxes are created for each member's modulus of elasticity. Each p-box is then discretized following the notation previously outlined as:

$$E_{iqL} = \overline{F_i}^{-1}\left(\frac{q-1}{z}\right) \quad (4)$$

$$E_{iqU} = \underline{F_i}^{-1}\left(\frac{q}{z}\right) \quad (5)$$

in order to generate z intervals:

$$\tilde{E}_{iq} = \left[E_{iqL}, E_{iqU}\right] \quad (6)$$

for each member i with uncertainty in its modulus of elasticity. This notation is slightly modified to:

$$\tilde{E}_{iq} = \left[x_{iqL}, x_{iqU}\right] * E \tag{7}$$

where a deterministic value E is multiplied by the coefficient interval $\left[x_{iqL}, x_{iqU}\right]$ containing all uncertainties for member *i*. The lower and upper interval coefficients are collected into vectors $x_{i(q_i)L}$ and $x_{i(q_i)U}$ respectively, where q_i are independently defined as possibilistic discretization levels for every element *i*: $q_i \in [1,2,\ldots,z]$. Therefore, the uncertain global stiffness matrix is:

$$\left[\tilde{\mathrm{K}}_G\right] = \left(\left[x_{1(q_1)L}, x_{1(q_1)U}\right]\right) * [\overline{\mathrm{K}}_1] + \left(\left[x_{2(q_2)L}, x_{2(q_2)U}\right]\right) * [\overline{\mathrm{K}}_2] + \left(\left[x_{N(q_N)L}, x_{N(q_N)U}\right]\right) * [\overline{\mathrm{K}}_N] \tag{8}$$

where the uncertain stiffness contribution of member *i* to the global stiffness matrix is:

$$\left[\tilde{\mathrm{K}}_i\right] = \left(\left[x_{i(q_i)L}, x_{i(q_i)U}\right]\right) * [\overline{\mathrm{K}}_i] \tag{9}$$

where $[\overline{\mathrm{K}}_i]$ is the deterministic contribution of the element stiffness matrix to the global structural stiffness for member *i*.

5. Uncertain Frequency Analysis

Two methods of polymorphic uncertainty frequency analyses have been previously developed utilizing imprecise probability (Modares and Bergerson 2012). Both polymorphic approaches, P-box Frequency Analysis (PFA) and Interval Monte-Carlo Frequency Analysis (IMFA), are developed from the isomorphic uncertainty frequency analysis entitled Interval Frequency Analysis (IFA) (Modares et al. 2006).

P-box Frequency Analysis is a combinatorial polymorphic uncertainty analysis which utilized p-box discretization of imprecise probability to generate imprecise probability bounds on all natural circular frequencies for any combination of uncertain system parameters. Because of the combinatorial nature of this method, this scheme is computationally feasible only for very small (limited number of members) or simple (few parameters controlling behavior) systems.

Interval Monte-Carlo Frequency Analysis is a simulation-based polymorphic uncertainty analysis which performs combinations of Monte-Carlo simulations for all uncertain parameters in the system in order to approach the exact imprecise probability bounds on each natural circular frequency of the system. For each realization of the simulation, random CDF values are independently selected for each uncertain parameter and an Interval Frequency Analysis is completed. After a sufficiently large number of realizations are observed, probabilistic information is generated for possibilistic bounds on each uncertain natural circular frequency. This method is computationally feasible.

6. Uncertain Response Spectrum Analyses Using Imprecise Probability

6.1. Computation of Uncertain Mode Shapes

In order to compute bounds on the uncertain mode shapes for a system, the following is developed. For one element (*i*), the uncertain stiffness contribution to the global stiffness matrix is redefined as:

$$[\tilde{K}_i] = [\overline{K}_{C_i}] + [\tilde{K}_{R_i}] \tag{10}$$

where $[\overline{K}_{C_i}]$ is the deterministic element central stiffness matrix contribution to the global stiffness matrix and $[\tilde{K}_{R_i}]$ is the element radial stiffness matrix contribution to the global stiffness matrix that contains all uncertainty for element *i*. The uncertain global stiffness matrix is likewise redefined as:

$$[\tilde{K}_G] = [\overline{K}_{G_C}] + [\tilde{K}_{G_R}] \tag{11}$$

where $[\overline{K}_{G_C}]$ is the deterministic central global stiffness matrix and $[\tilde{K}_{G_R}]$ is the non-deterministic radial global stiffness matrix, containing all uncertainty within the input parameters (stiffness matrices).

Utilizing the perturbation theory of a symmetric non-negative definite matrix, the equation of the j^{th} uncertain mode shape, $\{\tilde{\varphi}_j\}$, corresponding to j^{th} uncertain natural frequency, $\tilde{\omega}_j$, is (Modares & Mullen 2013):

$$\{\tilde{\varphi}_j\} = \{\varphi_j\} + \left([\Phi_j] * \left({\omega_j}^2 * [I] - [\Omega_j]^2\right)^{-1} * [\Phi_j]^T\right) * \left([M]^{-1/2} * [\tilde{K}_{G_R}] * [M]^{-1/2}\right) * \{\varphi_j\} \tag{12}$$

where and ω_j and $\{\varphi_j\}$ are the j^{th} deterministic natural circular frequency and mode shape computed using the deterministic central global stiffness matrix, $[\Phi_j]$ is the matrix containing all ordered mode shapes as columns with the j^{th} mode shape removed, or:

$$[\Phi_j] = [\varphi_1|\varphi_2| \dots |\varphi_{j-1}|\varphi_{j+1}| \dots |\varphi_{a-1}|\varphi_a] \tag{13}$$

and $[\Omega_j]$ is the diagonal matrix with all natural circular frequencies ordered along its main diagonal with the j^{th} row and column removed, or:

$$[\Omega_j] = \begin{bmatrix} {\omega_1}^2 & 0 & & \cdots & 0 & 0 & & \dots & 0 & 0 \\ 0 & {\omega_2}^2 & & \cdots & 0 & 0 & & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & & \cdots & {\omega_{j-1}}^2 & 0 & & \cdots & 0 & 0 \\ 0 & 0 & & \cdots & 0 & {\omega_{j+1}}^2 & & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & & \cdots & 0 & 0 & \cdots & & {\omega_{a-1}}^2 & 0 \\ 0 & 0 & & \cdots & 0 & 0 & \cdots & & 0 & {\omega_a}^2 \end{bmatrix} \tag{14}$$

Equation 12 may be reduced to:

$$\{\tilde{\varphi}_j\} = \left([I_a] + [C_j] * (\textstyle\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i])\right) * \{\varphi_j\} \tag{15}$$

where $[I_a]$ is an identity matrix of size equal to the system's active degrees of freedom, a, $[C_j]$ is the diagonal matrix defined as:

$$[C_j] = [\Phi_j] * \left(\omega_j{}^2 * [I] - [\Omega_j]^2\right)^{-1} * [\Phi_j]^T \tag{16}$$

$[E_i]$ is the diagonal matrix defined as:

$$[E_i] = \left(\frac{x_{irU} - x_{irL}}{2}\right) * [\mathrm{M}]^{-1/2} * [\overline{\mathrm{K}}_i] * [\mathrm{M}]^{-1/2} \tag{17}$$

and $\tilde{\varepsilon}_i$ is the radial interval of uncertainty defined as $\tilde{\varepsilon}_i \in [-1,1]$.

6.2. DETERMINATION OF UNCERTAIN MODAL RESPONSE

Considering a proportional excitation, $\{P(t)\}$, defined as:

$$\{P(t)\} = P(t) * \{p\} \tag{18}$$

where $\{p\}$ is the loading (directional) vector and $(P(t))$ is the (scalar) time-dependent magnitude, applied to a system with a known (or assumed) modal damping ratio, ζ_n, the deterministic maximum nodal displacement is computed using a known response spectra as:

$$\{u_{j_{max}}\} = D_{j_{max}} * \Gamma_j * \{\varphi_j\} \tag{19}$$

where $D_{j_{max}}$ is the (scalar) maximum modal coordinate taken from the response spectra, which has a finite value due to the existence of damping, at ω_j and Γ_j is the (scalar) time-independent modal participation factor, computed as:

$$\Gamma_j = \frac{\{\varphi_j\}^T * \{p\}}{\{\varphi_j\}^T * [\mathrm{M}] * \{\varphi_j\}} \tag{20}$$

When considering uncertainty within the response spectrum analysis, the uncertain maximum nodal displacement is:

$$\{\tilde{u}_{j_{max}}\} = \widetilde{D}_{j_{max}} * \tilde{\Gamma}_j * \{\tilde{\varphi}_j\} \tag{21}$$

where $\tilde{D}_{j_{max}}$ is the uncertain modal contribution factor determined by considering all possible response spectra values for the uncertain natural frequency bound by the interval $[\omega_{jrL}, \omega_{jrU}]$ and $\tilde{\Gamma}_j$ is the uncertain modal participation factor with interval bounds computed as:

$$\tilde{\Gamma}_j = \frac{\{\tilde{\varphi}_j\}^T * \{p\}}{\{\tilde{\varphi}_j\}^T * [\mathrm{M}] * \{\tilde{\varphi}_j\}} \tag{22}$$

Because the uncertain modal participation factor is dependent on the uncertain eigenvector, functional dependencies must be considered when evaluating the uncertain nodal displacement. Thus, Equation 21 may be expanding by substituting for $\tilde{\Gamma}_j$ and $\{\tilde{\varphi}_j\}$ as:

$$\{\tilde{u}_{j_{max}}\} = \{\tilde{D}_{j_{max}} * \frac{\{p\}^T * \{\varphi_j\} * [I_a] + \{p\}^T * \{\varphi_j\} * [C_j] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) + \cdots}{\{\varphi_j\}^T * [\mathrm{M}] * \{\varphi_j\} + \{\varphi_j\}^T * [\mathrm{M}] * [C_j] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) * \{\varphi_j\} + \cdots}$$

$$\frac{\{p\}^T * [C_j] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) * \{\varphi_j\} * [I_a] + \{p\}^T * [C_j] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) * \{\varphi_j\} * [C_j] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i])}{\{\varphi_j\}^T * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]^T) * [C_j] * [\mathrm{M}] * \{\varphi_j\} + \{\varphi_j\}^T * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]^T) * [C_j] * [\mathrm{M}] * [C_j] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) * \{\varphi_j\}} * \{\varphi_j\}\} \tag{23}$$

As with the deterministic analysis, only the maximum possible value of the uncertain maximum nodal displacement is of interest. Thus:

$$\{\tilde{u}_{j_{max}}\}_{max} = \{max(\tilde{D}_{j_{max}} * \frac{\{p\}^T * \{\varphi_j\} * [I_a] + \{p\}^T * \{\varphi_j\} * [C_j] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) \ldots}{\{\varphi_j\}^T * [\mathrm{M}] * \{\varphi_j\} + \{\varphi_j\}^T * [\mathrm{M}] * [C_j] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) * \{\varphi_j\} \ldots}$$

$$\frac{+\{p\}^T * [C_j] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) * \{\varphi_j\} * [I_a] + \{p\}^T * [C_j] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) * \{\varphi_j\} * [C_j] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i])}{+\{\varphi_j\}^T * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]^T) * [C_j] * [\mathrm{M}] * \{\varphi_j\} + \{\varphi_j\}^T * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]^T) * [C_j] * [\mathrm{M}] * [C_j] * (\sum_{i=1}^{n} \tilde{\varepsilon}_i * [E_i]) * \{\varphi_j\}} * \{\varphi_j\})\} \tag{24}$$

6.3. DETERMINATION OF UNCERTAIN GLOBAL RESPONSE

The maximum total system displacement is computed using the square root sum of squares (SRSS) as:

$$\{\tilde{U}_{max}\}_{max} = \sqrt{\sum_{j=1}^{a} \{\tilde{u}_{j_{max}}\}_{max}^2} \tag{25}$$

This final step concludes the steps of each simulation. All interval bounds and maximum values for uncertain outputs are stored for each simulation.

6.4. UNCERTAIN BOUNDS ON RESPONSE

For uncertain outputs for which interval bounds are computed during each simulation, these bounds generated during each simulation are ordered to generate p-boxes for these uncertain outputs. For uncertain outputs for which only a maximum value was computed during each simulation, these upper limits are used to create a probabilistic upper possibility bound on the output.

7. Uncertain Response Spectrum Analyses Using Imprecise Probability

7.1. EXAMPLE 1

The first example obtains the maximum uncertain dynamic responses of a two span two story steel (E = 200GPa & ρ = 7.85g/cm³) frame with 18 active degrees of freedom subjected to the Newmark, Blume and Kapur (NBK) design response spectra. The system has independent uncertainty in the modulus of elasticity of each member. Each member in the system has uncertainty represented by a p-box, with lower and upper bounds following normal distribution. For this system, the steel self-weight is ⅓% of the total mass of the system. Deterministic modal damping ratios of 2% are used for all modes for this system.

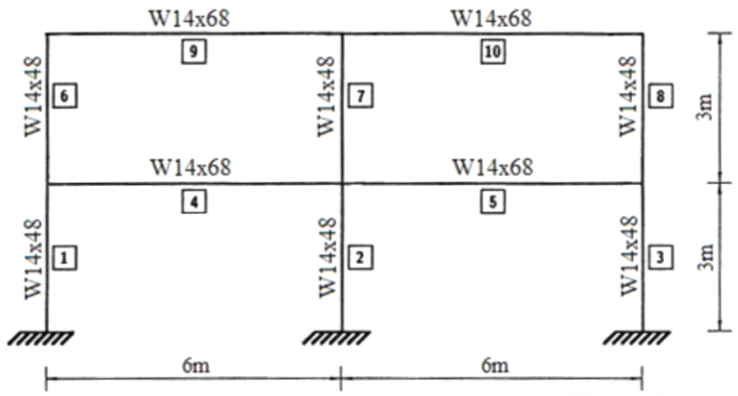

Figure 7. Two story, two span steel frame system.

The stiffness of each member is:

$$E_i: \tilde{\mu} = [0.95, 1.05] * E, \sigma = 0.0194 * E$$

CDF levels of ν and 1-ν, where ν = .005, are selected for truncating the infinite tails. The p-box for member *i* generated from the above uncertain modulus of elasticity is given below in Figure 8.

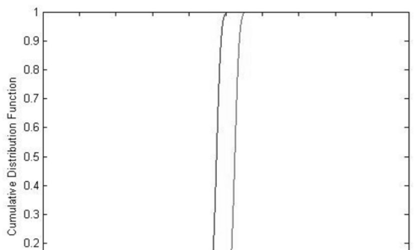

Figure 8. P-box for E_i.

7.1.1. *Determination of Uncertain Natural Circular Frequencies Using PFA*

The p-box above is discretized into five intervals (Eq. 4 & 5) and the results are given in the Table I.

Table I. Coordinates for the discretized p-box for member *i*	
E_i/E	
x_{iqL}	x_{iqU}
0.9000	1.0337
0.9337	1.0451
0.9451	1.0549
0.9549	1.0663
0.9663	1.1000

The discretized p-box for the modulus of elasticity for member *i* is given in Figure 9.

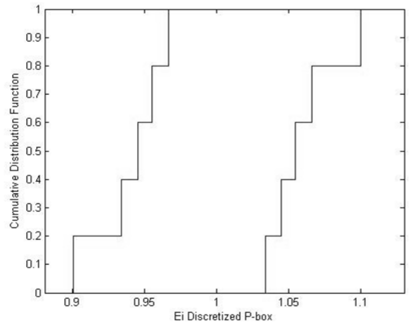

Figure 9. Discretized p-box for E_i.

Using PFA, upper and lower bounds on each natural circular frequency of the system are computed, and the data is condensed to five intervals. Table II shows the results for the fundamental natural circular frequency which is also depicted in Figure 10.

Table II. Condensed p-box coordinates for fundamental natural frequency using PFA	
$\tilde{\omega}_{1q}$	
ω_{1qL}	ω_{1qU}
4.4197	4.7798
4.4995	4.7902
4.5120	4.7998
4.5223	4.8114
4.5335	4.8861

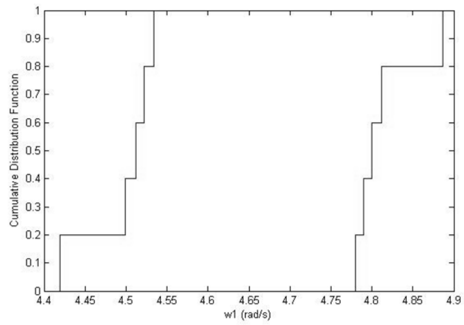

Figure 10. Condensed p-box for ω_1 from PFA.

7.1.2. *Alternate Determination of Uncertain Natural Circular Frequencies Using IMFA*

Alternatively, the problem is solved using 10^3 simulations. Upper and lower stiffness coefficients are generated for each member, and IMFA is used to obtain upper and lower bounds on all natural circular frequencies for each realization of the simulation. The results for the upper and lower bounds on all natural circular frequencies are condensed to five intervals. Table III shows the results for the uncertain bounds on the fundamental natural circular frequency.

Table III. Condensed p-box coordinates for fundamental natural circular frequency using IMFA, 10^3 simulations

$\tilde{\omega}_{1q}$	
ω_{1qL}	ω_{1qU}
4.4800	4.7601
4.5263	4.7699
4.5366	4.7776
4.5449	4.7855
4.5532	4.8190

The bounds on the fundamental natural circular frequency from both PFA (solid lines) and IMFA (dashed lines) are depicted in Figure 11. It is noted that the bounds on the fundamental natural circular frequency obtained from IMFA are inner-estimates of those from PFA. This has also been observed for higher order natural circular frequencies.

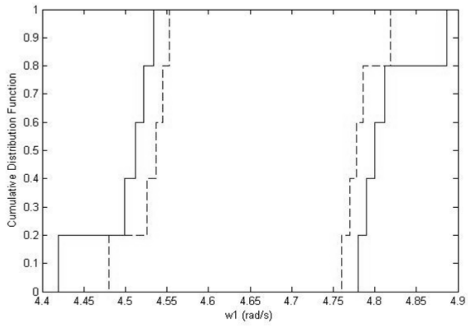

Figure 11. Condensed p-box for ω_1, with IMFA results represented by dashed lines.

7.1.3. *Determination of Maximum Uncertain Modal Responses and Global Responses Using IPRSA*

The results from IMFA are utilized for each uncertain natural circular frequency. For each realization of the simulation, the maximum uncertain maximum modal responses are computed using Equation 24. The maximum nodal responses are then computed by combining the maximum modal responses using SRSS method, as shown in Equation 25. The results of the maximum uncertain nodal responses are stored for each realization of the simulation. After 10^3 simulations are completed, the results for the maximum uncertain total response are organized to generate probabilistic information on the possibilistic upper bound for the maximum uncertain global response. The values of the maximum uncertain global response for the first and second story displacements corresponding to CDF values of 0, 0.5, and 1.0 are given in Table IV.

Table IV. Maximum uncertain and deterministic central (DC) horizontal story displacement values for specified CDF values

Story	Horizontal Displacement (m)		
	CDF=0	CDF=0.5	CDF=1.0
1	0.084224	0.085698	0.087199
1, DC	0.081941	0.08334	0.084763
2	0.16773	0.16956	0.17209
2, DC	0.16320	0.16488	0.16723

Figures 12 and 13 show the upper bounds on maximum uncertain horizontal displacement for the first story and second story, respectively.

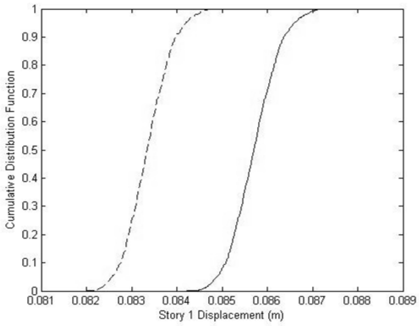

Figure 12. Upper bound on maximum uncertain horizontal displacement for first story from IPRSA, represented by the solid line, and the deterministic horizontal displacement of the first story for deterministic central RSA represented by the dashed line.

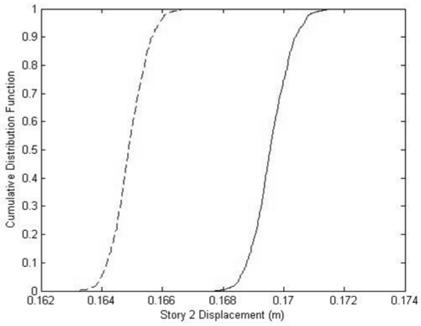

Figure 13. Upper bound on maximum uncertain horizontal displacement for second story from IPRSA, represented by the solid line, and the deterministic horizontal displacement of the second story for deterministic central RSA represented by the dashed line.

7.2. EXAMPLE 2

The second example problem obtains the maximum uncertain dynamic response of a truss system subjected to a heavy-side step loading function, with uncertainty in the system's damping ratios. Each member in the truss has the same cross section and material properties.

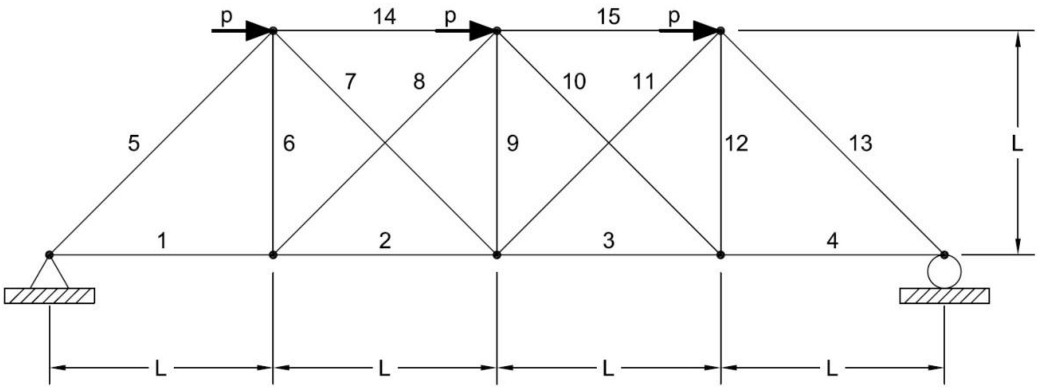

Figure 14. Truss subjected to heavy-side step loading function as shown.

The damping ratios for each natural frequency are independently defined as:

$$\zeta_n: \tilde{\mu} = [0.03, 0.07], \sigma = 0.008$$

7.2.1. *Determination of Maximum Uncertain Modal Responses and Global Responses Using IPRSA*
Because the system does not have any uncertainty in either the mass or stiffness properties, deterministic frequency analysis is completed prior to starting the simulation. Thus, deterministic natural circular frequencies and deterministic mode shapes are computed. Likewise, deterministic modal participation factors are also computed prior to beginning simulation (Eq. 20). The only portion of the modal response that contains uncertainty is in the dynamic response coordinate. For a heavy-side step load function, the deterministic dynamic response coordinate is computed as:

$$D_n = 1 + e^{\frac{-\zeta_n * \pi}{\sqrt{1-\zeta_n^2}}} \tag{26}$$

Because only the maximum nodal responses are of interest, for uncertain damping ratio, only the lower bounds on the uncertain damping ratios control. Thus:

$$\tilde{D}_{n_{max}} = 1 + e^{\frac{-\zeta_{n_{min}} * \pi}{\sqrt{1-\zeta_{n_{min}}^2}}} \tag{27}$$

For each realization of the simulation, a damping ratio is generated from the lower bound of imprecise probability for the uncertain damping ratio. The maximum dynamic response coordinate is then computed using Equation 27. The maximum modal responses are then computed using Equation 19. The maximum total response is then computed by combining the maximum modal responses using SRSS method, as shown in Equation 25. The results of the maximum uncertain nodal responses are stored for each realization of the simulation. After 10^4 simulations have been completed, the results are organized. The maximum horizontal displacements for the three top chord nodes corresponding to CDF values of 0, 0.5, and 1.0 are given in Table V.

Table V. Maximum uncertain and deterministic central (DC) horizontal displacement for three top chord nodes for specified CDF values

Node	Horizontal Displacement (U/(PL/AE))		
	CDF=0	CDF=0.5	CDF=1.0
Left	0.543113	0.559792	0.577631
Left, DC	0.527805	0.543511	0.560292
Middle	0.518790	0.534722	0.551762
Middle, DC	0.504168	0.519171	0.535200
Right	0.738803	0.761491	0.785758
Right, DC	0.717979	0.739345	0.762172

Figure 15 shows the upper bounds on maximum uncertain horizontal displacement for the three top chord nodes.

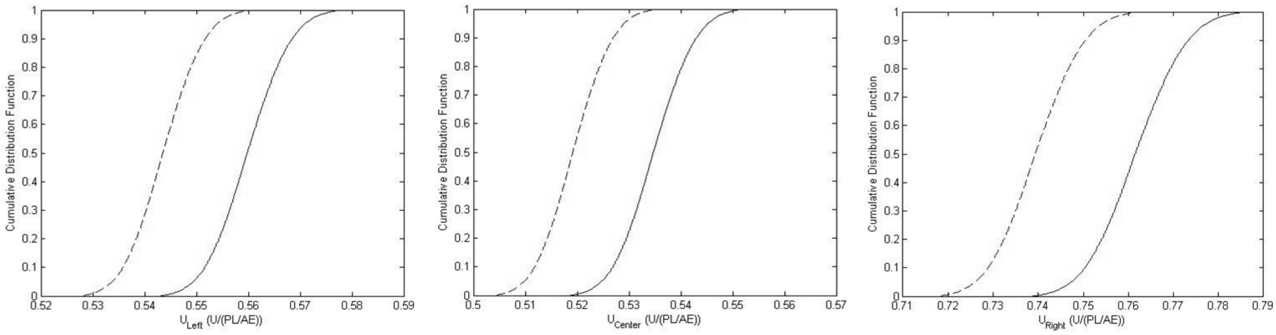

Figure 15. A, B, & C, upper bounds on maximum uncertain horizontal displacement and maximum pseudo-deterministic (central) displacements, represented by dashed lines, for left, center, and right top chord nodes, respectively.

8. Summary and Conclusions

Imprecise probability is one method of polymorphic uncertainty analysis that is capable of handling uncertainty in the data and model, as well as in their distributions. Two new methods for frequency analysis of structures with uncertain properties defined by independent p-boxes were developed. These developed methods allow for uncertainty in the stiffness matrix and can be extended to consider uncertainty in the mass matrix under the same framework.

P-box Frequency Analysis provides exact bounds on all natural frequencies, but is computationally feasible only for small systems. Interval Monte-Carlo Frequency Analysis provides a useful framework that is computationally feasible for any sized system that is also reliable and robust.

A method for Response Spectrum Analysis capable of handling uncertainty in the stiffness matrix was developed entitled Imprecise Probability Response Spectrum Analysis. This method utilizes simulation to obtain a computationally feasible scheme to generate probabilistic information on the maximum uncertain dynamic response of the system subjected to dynamic loading.

Numerical examples illustrating the application of the developed methods verified the computational feasibility of the developed methods. The simplicity of the proposed methods makes them useful for introducing uncertainty defined by imprecise probability into structural dynamics.

References

Augenbaugh, J. M., and Paredis, C. J. J. The Value of Using Imprecise Probabilities in Engineering Design, *Journal of Mechanical Design*, 128(4), 969-979, 2006.

Bellman, R. *Introduction to Matrix Analysis*. New York: McGraw-Hill Book Company, 1970. Print.

Berleant, D., and Goodman-Strauss, C. Bounding the Results of Arithmetic Operations on Random Variables of Unknown Dependency Using Intervals, *Reliable Computing*, 4(2), 147-165, 1998.

Ferson, S., and Donald, S. Probability Bounds Analysis, *Probabilistic Safety Assessment and Management*, Mosleh, A., and Bari, R.A. Eds., Springer-Verlag, New York, NY, 1203-1208, 1998.

Ferson, S., Kreinovich, V., Ginzburg, L., Myers, D., S., Sentz, K. Construction probability boxes and Dempster-Shafer structures, Tech. Rep. SAND2002-4015, Sandia National Laboratories 2003.

Modares, M., and Bergerson, J. Reliable Dynamic Analysis of Structures Using Imprecise Probability, *5th International Workshop on Reliable Engineering Computing*, 315-324, 2012.

Modares, M., and Mullen, R. Dynamic Analysis of Structures with Interval Uncertainty, *Journal of Engineering Mechanics*, ASCE, May ed., 0733-9399, 2013.

Modares, M., Mullen, R., L., Muhanna, R., L. Natural Frequencies of a Structure with Bounded Uncertainty, *Journal of Engineering Mechanics*, ASCE, December ed., 1363-1371, 2006.

Muhanna, R. L., and Mullen, R. L. Uncertainty in Mechanics Problems-Interval-Based Approach. *Journal of Engineering Mechanics*, June-2001, 557-566, 2001.

Oberkampf, W., Helton, J. and Sentz, K. "Mathematical Representations of Uncertainty," *AIAA Non-Deterministic Approaches Forum, AIAA 2001-1645*, Seattle, WA, April 16-19, 2001.

Williamson, R., and Downs, T. Probabilistic Arithmetic I: Numerical Methods for Calculating Convolutions and Dependency Bounds, *International Journal of Approximate Reasoning*, 4, 89-158, 1990.

Zhang, H., Mullen, R., L., Muhanna, R., L. Finite Element Structural Analysis using Imprecise Probabilities Based on P-Box Representation, *4th International Workshop on Reliable Engineering Computing*, 211-225, 2010.