Reliable Condition Assessment of Structures Using Uncertain or Limited Field Modal Data

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Abstract: In this paper a new method for reliable condition assessment and damage detection of structures is presented. The method uses a stochastic finite element analysis along with uncertain or limited field modal data (expressed by bounded random variables) for a structure for estimating any damage occurred to the structure. The basic steps in this new development is to (1) construct a finite element (FE) model, including stiffness and mass matrices, of the undamaged structure, (2) quantify the uncertainty in the measured modal data as bounded random variables, (3) perform Monte-Carlo simulations to obtain the FE model of existing structure through iterative optimization method used in estimating the stiffness of the damaged structure, (4) utilize the element stiffness matrices in each realization to identify the damage members based on the difference between a structural element stiffness for the "as built condition" and "damaged condition", and (5) determine the bounds on the location, as well as the extent of damage, that caused the degradation of the system. A numerical illustration is presented to demonstrate the capability of the method to detect the location and extent of the damage. It has been shown that, in the presence of uncertainty in, or with limited information on, the modal data, the method is capable of determining the bounds on location and extent of damage.

Keywords: Condition Assessment of Structures; Modal Data; Finite Element Model; Monte Carlo Simulation; Structural Degradation.

1. Introduction

Structural Health Monitoring (SHM) techniques are employed to determine the prevalence, location, and extent of damage using condition assessment methods and available measured structural responses. SHM systems have increasingly been used to assess the condition of infrastructure systems. This is mainly because of (1) the capability of SHM systems to provide continuous monitoring processes for detecting anomalies and distress conditions, and (2) the advancement in, and the availability of, sensor technologies as well as effective wireless data transfer procedures at a relatively low cost. However, most condition assessment methods have shortcomings particularly in conjunction with field measurements because: (1) the available data is limited, and (2) the inherent uncertainties in both experimental data and analytical methods lead to limited information about the condition of structures.

Condition assessment methods are mainly based on nondestructive damage identification techniques, which are categorized as local (e.g. ultrasonic and X-ray methods) and global (e.g. pattern recognition and vibration-based methods). Local damage identification techniques require that the general area, and the proximity of damage location, are known in advance and be accessible for test and data acquisition. This may not always be feasible for most cases in civil and airframe structural systems. Conversely, most global damage identification techniques are capable of detecting damage with limited data as well as no requirement of prior knowledge and accessibility of damage vicinity.

One of the common global damage identification techniques is the vibration-based approach, which uses the measured modal data, along with an analytical scheme, to detect damage. In general, the vibration-based approach can be grouped into non-model based (also known as pattern recognition) and model based methods. Non-model based methods can only detect damage location (Nair et al. 2006); whereas model based methods can determine both location and extent of damage (Huang et al. 2012). The underlying concept in the vibration-based damage identification is the fact that the damage alters the structure's physical properties (e.g. mass, damping, and stiffness), and consequently causes detectable changes in its modal data (e.g. natural circular frequencies and mode shapes). However, the available measured modal data is often from accelerometer readings; and as such, the information may be subject to uncertainties, and even errors. These stem from the sensor problem (e.g. lack of proper calibration), human errors or mistakes in installation practices. Furthermore, even with adequate equipment, and exercising proper care and conducting a rigorous data quality assurance procedures, the modal data compiled is generally very limited and may not be representative of the dominant mode of vibration. Therefore, the application of most state-of-the-art vibration-based damage identification techniques, when used in conjunction with field measurements, may produce limited results. These issues can be attributed to 1) measurement noise 2) incompleteness of modal data, and 3) uncertainties inherent in data and analytical methods.

In the study reported herein, a new vibration-based probabilistic method for condition assessment of structures through a Monte-Carlo simulation procedure is developed and presented. The method is capable of determining the bounds on extent and location of damage. The method will hereafter be referred to as the Stochastic Structural Condition Assessment (SSCA). Specifically, the method uses a stochastic Finite Element (FE) analysis utilizing uncertain or limited field modal data regarding the modal frequency response of the structure in assessing its condition. The uncertain or limited modal data is expressed by a series of bounded random variables in the model to arrive at an estimate for the condition of the structure. The following presents the steps involved in this method. (1) A Finite Element (FE) model, including stiffness and mass matrices, of the undamaged structure is constructed based on engineering drawings and available information or from site investigations and direct field measurements. The system stiffness matrix corresponding to this model represents the system in an undamaged condition. (2) The uncertainty in the available measured modal data is quantified as bounded random variables using statistical methods. (3) For each realization of Monte-Carlo simulations, the FE model of existing structure is obtained through an iterative optimization method used in estimating the stiffness of the damaged structure. (4) For each realization, the element stiffness matrices are used to identify the damage members based on the difference between the structural element stiffness for the "as built condition" and "damaged condition". (5) Using the Monte-Carlo simulation results, the bounds on the location as well as the extent of damage (that caused the degradation of the system) are determined.

To demonstrate its applicability, the new development is applied, as one of the numerical examples, to the structural health monitoring benchmark problems given jointly by the International Association for Structural Control and the dynamics committee of the American Society of Civil Engineers (IASC-ASCE). The method has been demonstrated to accurately determine the bounds on location and extent of damage in that model structure as described later in this paper.

2. An Overview of Development of Damage Detection through SHM

In recent years, extensive research has been conducted on structural health monitoring (SHM) applications. Most such research, and respective models that have been developed, are based on data compiled on a host of structural parameters including the modal frequencies and vibration characteristics of the structure. The accuracy of the results from any SHM, to a large degree, depend on (1) the adequacy of the compiled data; and (2) the fact that whether the compiled modal data is complete, accurate and is triggering the specific degrees of freedom that are most representative of the damaged components. As expected, an accurate result on the condition of the structure depends on accurate and complete modal data. However, the challenge in obtaining an accurate or a reasonable estimate for the condition of the structure is mainly in cases where the modal data is incomplete or is subject to uncertainties. In this section, we start by providing a summary of pertinent studies related to damage detection of structures using SHM as a baseline in introducing our model and its significance in offering reasonable results based on uncertain and/or limited modal data.

Doebling et al. (1996) presented an extensive review of vibration-based damage detection methods. Salawu (1997) presented an extensive review of structural damage detection through frequency changes and discussed the use of natural frequency as a diagnostic parameter in structural assessment procedures using vibration monitoring. Salawu suggested that natural frequency changes alone may not be sufficient for a unique identification of the location of structural damage. This is because cracks associated with similar crack length but at two different locations may cause the same amount of frequency change.

Friswell (1995) presented a FE model updating in structural dynamics which covers different aspects of model preparation and data acquisition necessary for updating. Sohn et al. (1997) presented a Bayesian probabilistic approach for structural damage detection and considered the measurement noise and error associated with the structural and analytical modeling. Messina et al. (1998) proposed methods for structural damage detection and estimating the size of defects in a structure based on the sensitivity of the frequency of each mode to damage in each location by introducing a correlation coefficient. Their approach offers the practical attraction of only requiring measurements of the changes in a few structure’s natural frequencies. Shi et al. (2000) extended the damage localization method based on study by Messina et al. (1998) and used incomplete mode shape instead of modal frequency in their modeling. In their method, the damage sites are localized first by using incomplete measured mode shapes. Then the damage site and extent are detected using measured natural frequencies, with a higher accuracy than using mode shapes. Ren and De Roeck (2002) proposed a damage identification technique from the FE model using frequencies and mode shape change. Their method is applied at an element level with a conventional FE model. The element damage equations have been established through the eigenvalue equations that characterize the dynamic behavior.

Kim and Stubbs (2003) presented a method to locate and quantify a crack in beam-type structures by using changes in a few natural frequencies. Crack location and crack size estimation models were formulated by relating fractional changes in modal energy to changes in natural frequencies. Moreover, Estes et al. (2003) proposed a general approach for using field information to update the reliability of a structure. Lee et al. (2005) presented a neural network-based technique for element level damage assessment of structures using the modal properties. In their method, the differences (or the ratios) of the mode shape components between the “before” and “after” damage conditions are used as the input to the neural networks; since they are less sensitive to the modeling errors than the mode shapes. Yuen et al. (2006) presented a methodology for Bayesian structural model updating using noisy incomplete modal data corresponding to natural frequencies and partial mode shapes of some of the modes of a structural system. Their method does not require matching measured modes with corresponding modes from the structural model. It uses an iterative scheme involving a series of coupled linear optimization problems in order to find the most probable values of model parameters.

Beck (2010) presented a rigorous framework for system identification based on probability logics. This framework uses probability as a multi-valued propositional logic for plausible reasoning where the probability of a model is a measure of its relative plausibility within a set of models. In his work, Bayes' Theorem is used to update the relative plausibility of each model in a group instead of using system data to estimate the model parameters. Fan et al. (2011) presented an extensive review on modal parameter-based damage identification methods and a comparative study based on FE model. Huang et al. (2012) proposed a probabilistic damage detection approach that is based on modal parameters extracted from ambient vibration responses. This method uses a Bayesian model updating and the damage index method to detect the damage.

Wang and Li (2012) presented a new method for damage localization and severity estimation based on the employment of modal strain energy. In their work, an iterative modal strain energy is developed based on the fact that only the true damage scenario can lead to the combinations of frequency changes measured from the damaged structure. However, throughout their work, the presence of uncertainties as well as incompleteness of information is not considered. Moreover, their proposed damage detection scheme uses a combinatorial approach that is not computationally feasible for a large structure. The method of SSCA, presented in this paper, enhances Wang and Li's method (2012) by (1) considering uncertainties in the modal data, (2) performing Monte-Carlo simulations and (3) developing a numerical optimization scheme for computational feasibility.

3. Methodology - Stochastic Structural Condition Assessment (SSCA)

The general algorithm for the method of Stochastic Structural Condition Assessment (SSCA) consists of the following steps:

I. Construct initial FE model
   - Model the intact structure based on design drawings and available information and/or field measurements (either to complement the existing information or as a stand-alone set of information).
   - Determine the intact structure's stiffness and mass matrices.
   - Determine the intact structure's modal characteristics (e.g. natural circular frequencies and mode shapes)

II. Determine the uncertainty in the modal data
   - Collect a series of sensor-based modal measurements (e.g. natural frequencies using accelerometers).
   - Quantify the uncertainties inherent in the measured data as bounded random variables using statistical methods.

III. Perform Monte-Carlo simulations for stochastic condition assessment of the structure.

In each realization step:

- Determine the FE model of damaged structure by obtaining structure's stiffness matrix using modal data and intact model through an iterative optimization scheme.
- Determine the extent and location of damage based on changes between the structural element stiffness for the "as built condition" and "damaged condition".
- Determine the bounds on both location and extent of damage that caused the degradation of the system (using the Monte-Carlo simulations results).

3.1. Initial FE Model

The initial FE model of the structure is constructed based on design drawings and available information and/or field measurements. Global stiffness matrix and mass matrix are determined in order to calculate the modal characteristic (natural circular frequencies and their corresponding mode shapes) of the structure using the generalized eigenvalue problem as:

$$K_u \Phi_{ui} = \omega_{ui}^2 M_u \Phi_{ui} \qquad i = 1, \dots, n \tag{1}$$

where, K_u and M_u are the global stiffness and mass matrices of the undamaged structure respectively; ω_{ui} and Φ_{ui} are the i^{th} natural circular frequency and its corresponding mode shape for the undamaged structure; and n is the number of active degrees-of-freedom in the undamaged structure.

3.2. Quantification of Uncertainty in the Measured Modal Data

The presence of uncertainties and incompleteness in the measured modal data is expressed by bounded random variables using statistical methods. For example, the bounded random variables for natural circular frequencies are obtained as intervals of form:

$$\widetilde{\omega}_{di} = [\mu - \delta, \mu + \delta] \qquad i = 1, \dots, n \tag{2}$$

where, $\widetilde{\omega}_{di}$ is the bounded random variable for the i^{th} measured natural frequency of the damaged structure, and δ is the half-width of the interval that may be determined using expert opinion and/or engineering judgment.

3.3. Monte-Carlo Simulations for SSCA

3.3.1 *FE Modeling for Damaged Structure for Each Realization*

In each realization, the generalized eigenvalue problem for the damaged structure is:

$$K_d \Phi_{di} = \omega_{di}^2 M_d \Phi_{di} \qquad for\ i = 1, \dots, m \tag{3}$$

where K_d is the unknown global stiffness matrix and M_d is the mass matrix; ω_{di} is a realization of the ith random natural circular frequency ($\omega_{di} \in \widetilde{\omega}_{di}$) , Φ_{di} is the ith mode shape of the damaged structure, and $m \leq n$ is the total number of measured mode shape of the damaged structure. Since local damage does not have significant effects on changing the structure's mass, it is assumed that $M_u = M_d$.

Considering the change between undamaged and damaged stiffness matrices, the global stiffness matrix of damaged structure is (Eq.(4) adopted from Wang and Li 2012):

$$K_d = K_u - \sum_{j=1}^{e} \alpha_j K_j \tag{4}$$

where, e is the total element number, K_j is the global stiffness matrix of the jth element; and α_j is the *extent of damage* (damage ratio) of the corresponding element. The extent of damage is the fractional change in stiffness of the element $\alpha_j = [0,1]$ in which $\alpha_j = 0$ implies no damage and $\alpha_j = 1$ implies completely damaged.

Pre-multiplying Eq. (1) by $(\Phi_{di})^T$ and Eq. (3) by $(\Phi_{ui})^T$ and using the properties of symmetric matrices $(\Phi_{di}{}^T K_d \Phi_{ui} = \Phi_{ui}{}^T K_d \Phi_{di})$, the relationship between global undamaged and global damaged stiffness is:

$$(\Phi_{ui})^T K_d \Phi_{di} = \frac{\omega_{di}^2}{\omega_{ui}^2} (\Phi_{ui})^T K_u \Phi_{di} \tag{5}$$

Combination of Eq. (4) and Eq. (5) yields:

$$\sum_{j=1}^{n} \alpha_j (\Phi_{ui})^T K_j \Phi_{di} = (1 - \frac{\omega_{di}^2}{\omega_{ui}^2})(\Phi_{ui})^T K_u \Phi_{di} \tag{6}$$

in which, $C_i = (\Phi_{ui})^T K_u \Phi_{di}$ is the mixed modal strain energy of the undamaged and damaged structure for the ith mode, and $C_{j,i} = (\Phi_{ui})^T K_j \Phi_{di}$ is the corresponding modal strain energy between the undamaged and damaged structure for the ith mode of the jth element, respectively. Using b_i to represent the right side of Eq. (6), it is simplified as:

$$\sum_{j=1}^{e} \alpha_j C_{j,i} = b_i \tag{7}$$

Eq. (7) is a rank-deficient system of equations. This is because α_j is unknown for all elements, and in general, the number of equations are limited to the available measured natural frequencies and mode shapes of structure. In order to solve Eq. (7), an optimization scheme is used to determine all α_j as explained below.

3.3.2 *Optimization Scheme*

The optimization scheme used in SSCA is capable of producing results with information available for as limited as one mode for both damaged and undamaged structure. In each realization, a least square approach is used to compute all α_j which must satisfy the constraints (as explained later). The problem is a linear constrained optimization solved by the MATLAB Optimization Toolbox using the LSQNONNEG routine. This routine utilizes a least square approach with non-negativity constraints. Eq. (7) is stated as:

$$\min_x \|Cx - b\|_2^2 \text{ , } where\ x \geq 0 \tag{8}$$

where, x is the unknown vector of size $(1 \times e)$, which represents the extent of damage for all e elements. The objective is to minimize the Euclidian norm of $(Cx - b)$ subject to $x \geq 0$ and also,$C \in \mathbb{R}$, $b \in \mathbb{R}$.

3.3.3 *Identification of Location and Extent of Damage*

In each realization, all values of α_j , which represent the extent of damage for a specific element, are computed. The index of this variable (j) represents the location of damage and the value of α_j represents and extent of damaged. For example, $\alpha_3 = 0.35$ means that extent of damage for element 3 is 35% (i.e, the loss of stiffness comparing to original stiffness of this element is 35%).

3.3.4 *Bounds on Location and Extent of Damage*

Using the Monte-Carlo simulation results, the bounds on both location and extent of damage (that caused the degradation of the system) are determined.

4. Numerical Example

SSCA is applied to a structural health monitoring benchmark model problem given jointly by the International Association for Structural Control and the dynamics committee of the American Society of Civil Engineers (IASC-ASCE). Upon conducting the analysis, the results show that SSCA is able to successfully determine the bounds on both location and extent of damage in the model structure.

The IASC–ASCE task group on SHM developed a series of benchmark SHM problems (Johnson et al. 2004) for which the damage is known. The benchmark problem investigated in this paper is a four-story two-bay by two-bay steel braced frame. It has a 2.5m × 2.5m plan, and is 3.6m tall (Figure 1).

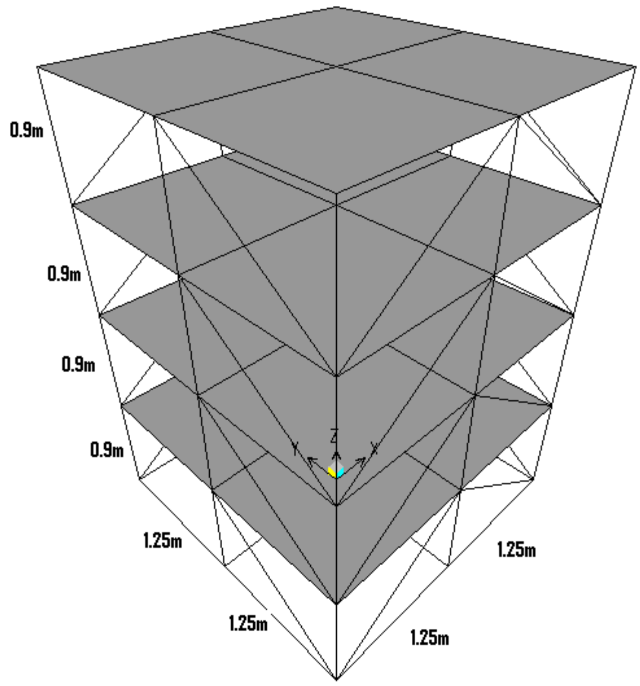

Figure 1. Diagram of the analytical model (adopted from Johnson et al. 2004).

The structural elements are hot-rolled steel with a nominal yield stress 300 MPa. Table I shows the properties of structural elements. The analytical model is a 12 DOF system and the structure is assumed to act as a shear building (it constrains all motion except two horizontal translations and one rotation per floor, i.e. 3 DOF per floor). The columns and floor beams are modeled as Euler–Bernoulli beams in the FE analysis. The braces are bars with no bending stiffness.

Table I. Properties of structural members			
Property	*Columns*	*Floor beams*	*Braces*
Section type	B100x9	S75x11 0.15	L25x25x3
Cross-sectional area A (m^2)	1.133x 10^{-3}	1.43x 10^{-3}	0.141x 10^{-3}
Moment of inertia (strong direction) I_y (m^4)	1.97x10^{-6}	1.22x10^{-6}	0
Moment of inertia (weak direction) I_z (m^4)	0.664x10^{-6}	0.249x10^{-6}	0
St. Venant torsion constant J (m^4)	8.01x10^{-9}	38.2x10^{-9}	0
Young's modulus E (Pa)	2x10^{11}	2x10^{11}	2x10^{11}
Shear modulus G (Pa)	*E*/2.6	*E*/2.6	*E*/2.6
Mass per unit volume ρ (kg/m^3)	7,800	7,800	7,800

Table II shows the horizontal story stiffness of undamaged 12 DOF model.

Table II. Horizontal story stiffness (MN/m) of undamaged model.		
Story	DOF	Undamaged Stiffness
1	x	106.60
1	y	67.90
1	θ	232.02
2	x	106.60
2	y	67.90
2	θ	232.02
3	x	106.60
3	y	67.90
3	θ	232.02
4	x	106.60
4	y	67.90
4	θ	232.02

Damage Pattern. In this example, the investigated damage pattern case considers no stiffness in all braces of the first and third stories (i.e. all braces in first and third stories are removed).

Modal Data. The reported deterministic natural circular frequencies of the damaged structure in the Benchmark problem are considered to be the mean values. In order to quantify the uncertainty present in the modal data, the bounded random variable for each natural circular frequency is determined using Eq. 2 considering half-width to be 5% of its mean value (i.e. $\delta = 0.05\mu$). These bounds of natural circular frequencies are then used for the Monte-Carlo simulation performed. Table III shows the mean values as well as bounds of natural circular frequencies.

Table *IV* shows the mean values (as reported in the Benchmark problem) and bounds (as calculated using SSCA) for the loss of horizontal story stiffness.

Table III. Mean values and bounds on natural circular frequencies of the damaged structure.

Mode	Mean (rad/s)	Lower and Upper Bounds (rad/s)
1	36.63	[34.80 , 38.46]
2	59.80	[56.81 , 62.79]
3	69.93	[66.43 , 73.43]
4	93.80	[89.11 , 98.49]
5	156.93	[149.08 , 164.77]
6	180.84	[171.80 , 189.89]
7	227.95	[216.55 , 239.35]
8	261.66	[248.58 , 274.74]
9	295.71	[280.93 , 310.50]
10	344.06	[326.86 , 361.27]
11	407.50	[387.13 , 427.88]
12	466.67	[443.34 , 490.00]

Table IV. Exact values and bounds of percent loss in horizontal story stiffness.

Story	DOF	Exact	Lower and Upper Bounds
1	x	45.24%	[23.97% , 50.49%]
1	y	71.03%	[70.99% , 93.41%]
1	θ	64.96%	[4.59% , 79.16%]
2	x	0	[0 , 9.57%]
2	y	0	[0 , 0]
2	θ	0	[0 , 0]
3	x	45.24%	[26.82% , 50.57%]
3	y	71.03%	[49.32% , 72.56%]
3	θ	64.96%	[39.25% , 65.33%]
4	x	0	[0 , 9.67%]
4	y	0	[0 , 11.95%]
4	θ	0	[0 , 32.27%]

Observations. The results of benchmark problem show that SSCA is capable of determining the bounds on both location and extent of damage. It also shows that with considering uncertainty in the measured modal data, 1) more structural members may have experienced damage and, 2) higher extent of damage may have occurred.

5. Features and Limitations of the Study

Using SSCA, the presence of uncertainty and incompleteness in the measured modal data is quantified as bounded random variables through application of statistical methods. Moreover, SSCA applies a numerical optimization scheme for obtaining the location and extent of damage in the presence of limited data.

The limitations of SSCA are as follows.

Both modal data of undamaged and damaged structure are required in the SSCA method. Although modal data for existing structures can be measured using sensors, the modal data for undamaged structures may not be available. And as such, construction of FE model based on design drawings and available information may cause more uncertainties. Furthermore, quantification of uncertainties in the measured data as bounded random variables is based on available data or expert opinion (e.g. 5%, 10%, etc.) that may lead to different bounds on both location and extent of damage. These limitations may result in additional errors in estimating the condition of a structure.

6. Summary and Conclusions

Using A new method for damage detection and assessment of structures using uncertain modal data, referred as Stochastic Structural Condition Assessment (SSCA), is developed.

SSCA applies finite element analyses and an optimization scheme along with Monte-Carlo simulation for more accurately assessing the condition of a structure with uncertain or limited data.

The main conclusions of the study are:

1. In the presence of uncertainty in the measured modal data, SSCA is capable of determining the bounds on both location and extent of any possible damage based on the change of structural stiffness occurred from the “as built condition” to the “damaged condition.”
2. Numerical illustrations show that SSCA, because of consideration of uncertainties, is capable of identifying the bounds on location and extent of damage more precisely.

References

Doebling, S.W., Farrar, C.R., Prime, M.B. and Shevitz, D.W. Damage Identification and Health Monitoring of Structural and Mechanical Systems from Changes in their Vibration Characteristics: A Literature Review. *Los Alamos National Laboratory report*,(LA-13070-MS), 1996.

Estes, A. C., Frangopol, D. M., and Foltz, S. D. Updating the reliability of engineering structures using visual inspection results. *Application of Statistics and Probability in Civil Engineering*, Der Kiureghian, Madanat and Pestana (eds) Millpress, Rotterdam, ISBN 90 5966 004 8, 1087-1092, 2003.

Fan, W. and Qiao, P.Z. Vibration-based Damage Identification Methods: A Review and Comparative Study. *Structural Health Monitoring*, 10, 83–111, 2011.

Friswell MI, Mottershead JE. Finite element model updating in structural dynamics. *Dordrecht, The Netherlands: Kluwer Academic*, 1995.

Haldar, A., and Mahadevan, S. Probability, Reliability and Statistical Methods in Engineering Design, *John Wiley and Sons, New York, NY*, 2000.

Huang Q, Gardoni P, Hurlebaus S. A probabilistic damage detection approach using vibration-based nondestructive testing. *Structural* 38, 11-21, 2011.

Johnson, E. A., Lam, H. F., Katafygiotis, L. S., and Beck, J. L. Phase I IASC-ASCE Structural Health Monitoring Benchmark Problem Using Simulated Data. *Journal of Engineering Mechanic*;130:3-15, 2004.

Kim, J.T. and Stubbs, N. Crack detection in beam-type structures using frequency data. *Journal of Sound and Vibration*, 259, 145–160, 2003.

Lee, J.J., Lee, J.W., Yi, J.H., Yun, C.B. and Jung, H.Y. Neural networks-based damage detection for bridges considering errors in baseline finite element models. *Journal of Sound and Vibration*, 280, 555–578, 2005.

Messina, A., Williams, E.J. and Contursi, T. Structural damage detection by a sensitivity and statistical-based method. *Journal of Sound and Vibration*, 216, 791–808, 1998.

Ren, W.X. and De Roeck, G. Structural damage identification using modal data. I: Simulation verification. *Journal of Structural Engineering-ASCE*, 128, 87–95, 2002.

Salawu, O.S. Detection of structural damage through changes in frequency: A review. *Engineering Structures*, 19, 718–723, 1997.

Shi, Z.Y., Law, S.S. and Zhang, L.M. Damage localization by directly using incomplete mode shapes. *Journal of Engineering Mechanics-ASCE*, 126, 656–660, 2000.

Wang, S., and Li, H. Assessment of structural damage using natural frequency changes. *Acta Mech. Sin.*;28(1):118-127, 2012.